DEPTH-ZERO BASE CHANGE FOR UNRAMIFIED $U(2,1)$

JEFFREY D. ADLER AND JOSHUA M. LANSKY

Abstract. We give an explicit description of $L$-packets and quadratic base change for depth-zero representations of unramified unitary groups in two and three variables. We show that this base change is compatible with unrefined minimal $K$-types.

1. Introduction

Given a finite Galois extension $E/F$ of finite, local, or global fields and a reductive algebraic $F$-group $G$, “base change” is, roughly, a (sometimes only conjectural) mapping from representations of $G = G(F)$ to those of $G(E)$. When $F$ is finite, or when $F$ is local and $G = GL(n)$, then this mapping is the Shintani lifting (as introduced in [25] and extended in [17], [16], and [13] for finite groups).

Correspondences like base change that are associated to the Langlands program can be difficult to describe explicitly, even in cases where they are known to exist. Bushnell and Henniart [5, 6, 8, 7] are remedying this situation for base change for $GL(n)$ over local fields. Analogously, Silberger and Zink [26] have made the Abstract Matching Theorem [11, 22, 2] explicit for depth-zero discrete series representations.

Suppose that $F$ is a $p$-adic field of odd residue characteristic. If $E/F$ is quadratic, and $G$ is a unitary group in three variables defined with respect to $E/F$, then Rogawski [23] has shown that a base change lifting exists, and has derived some of its properties. Our goal in this paper is to describe base change explicitly for depth-zero representations in the case where $E/F$ is unramified. Depth-zero base change is particularly interesting because it should be closely related to base change for finite groups. See [19] for an exploration of another special case of this phenomenon.

In order to apply a technical lemma (Cor. 2.6), we will assume that the order $q$ of the residue field $k_F$ of $F$ is at least 59. From the lemma, character identities can be verified by evaluation at “very regular” elements. At such elements, character values are particularly easy to compute. Without the lemma, the verification of these identities involves evaluation at more general elements. Character values at these elements can be computed, but are far more complicated.

Note that we assume that $F$ has characteristic zero only so that we can apply results of Rogawski [23]. Our calculations apply equally well if $F$ is a function field of odd residue characteristic.


Key words and phrases. $p$-adic group, base change, Shintani lift, $L$-packet, Langlands functoriality, unitary group.

The first-named author was partially supported by the National Security Agency (#MDA904-02-1-0020).
Conjecturally, one should be able to determine the depth of the representations in an \(L\)-packet from the associated Langlands parameter. Thus, liftings that arise from the Langlands correspondence should preserve depth, if depth is normalized correctly. In particular, depth-zero representations should go to depth-zero representations. We assume this throughout for base change and for endoscopic lifting from \(U(1,1) \times U(1)\) to \(U(2,1)\).

In \(\S 2\), we present our notation, review the general notion of Shin tani lifting, describe how it applies to the representations of certain finite reductive subquotients of \(G\), and list all of the representations of \(G\) of depth zero. In \(\S 5\), we determine the base change lift of each of these packets. In \(\S 6\) we examine the relationship between base change and \(K\)-types, as defined by Bushnell-Kutzko \cite{9} and as described by Moy-Prasad \cite{21} or Morris \cite{20}. Recall that a (minimal) \(K\)-type (or simply a “type”) of depth zero is a pair \((G_x, \sigma)\), where \(G_x\) is a parahoric subgroup of \(G\), and \(\sigma\) is the inflation to \(G_x\) of an irreducible cuspidal representation of the finite reductive quotient \(G_x\) of \(G_x\). Since all of the data in this definition can be lifted in a natural way to similar data for \(\tilde{G} = \tilde{G}(E)\), we have a natural notion of base change for depth-zero types. Under the above assumption on the residue characteristic of the \(p\)-adic field \(F\), we show that base change for depth-zero types is compatible with base change for representations (actually, \(A\)-packets of representations):

**Theorem 1.1.** Suppose \(\Pi\) is a depth-zero \(A\)-packet for \(G\), let \(\tilde{\pi}\) denote the base-change lift of \(\Pi\), and let \(\pi \in \Pi\). Suppose \((G_x, \text{infl}(\sigma))\) is a type contained in \(\pi\). Then \(\tilde{\pi}\) contains \((\tilde{G}_x, \text{infl}(\tilde{\sigma}))\), where \(\tilde{\sigma}\) is the base-change lift of \(\sigma\) from \(G_x\) to \(\tilde{G}_x\).

Note that the pair \((\tilde{G}_x, \text{infl}(\tilde{\sigma}))\) contains a type upon restriction to some parahoric subgroup of \(\tilde{G}_x\). Thus, either it is itself a type, or it carries more information than a type.

In \(\S 7\), we state a formula for the character of an induced representation. The formula itself is not new, but we need to assert that it holds for representations of \(\tilde{G}\) that are not necessarily connected.

In order to describe explicit base change for all representations of \(U(2,1)\) (not just of depth zero), one needs to understand depth-zero base change not just for \(U(2,1)\) but for unitary groups in two variables as well. We deal with this briefly in \(\S 3\).

We thank Robert Kottwitz, Jonathan Rogawski, A. Raghuram, David Pollack, Stephen DeBacker, and a referee for helpful communications.

## 2. Preliminaries

### 2.1. General notation and facts.

For any nonarchimedean local field \(F\), let \(\mathcal{O}_F\) denote its ring of integers, \(p_F\) the prime ideal in \(\mathcal{O}_F\), and \(k_F = \mathcal{O}_F/p_F\) the residue field. For any abelian extension \(E/F\), let \(\omega_{E/F}\) denote the character of \(F^\times\) arising via local class field theory.

We will use underlined letters to denote algebraic groups and will drop the underlining to indicate the corresponding groups of rational points. Given an algebraic \(F\)-group \(G\) and a finite extension \(E/F\), let \(\tilde{G} = \text{Res}_{E/F}(G)\), where \(\text{Res}_{E/F}\) denotes
Table 1. Depth-zero \(A\)-packets for \(U(2,1)\), and their base change lifts. To obtain \(L\)-packets, omit the representations marked with an asterisk (*)..

<table>
<thead>
<tr>
<th>(A)-packet*</th>
<th>Base change lift</th>
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<tbody>
<tr>
<td>{\text{Ind}^G_{\tilde{G}}\lambda} (\text{Ind}^G_{\tilde{G}}\lambda \text{ irreducible and } \text{Ind}^G_{\tilde{G}}\lambda \text{ irreducible}) \ (\S 4.1)</td>
<td>\text{Ind}^G_{\tilde{G}}\lambda \ (\S 5.1)</td>
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<tr>
<td>{\text{Ind}^G_{\tilde{G}}\lambda} (\text{Ind}^G_{\tilde{G}}\lambda \text{ irreducible and } \text{Ind}^G_{\tilde{G}}\lambda \text{ reducible}) \ (\S 4.1)</td>
<td>\text{Ind}^G_{\tilde{G}}\left( (\lambda_1 \lambda_2) \cdot \left</td>
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<tr>
<td>{\psi} (one-dimensional) \ (\S 4.1)</td>
<td>\psi \ (\S 5.1)</td>
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<td>{\text{St}_{\tilde{G}}(\psi)} \ (\S 4.1)</td>
<td>\text{St}_{\tilde{G}}(\psi) \ (\S 5.1)</td>
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<td>{\pi_1(\lambda), \pi_2(\lambda)} \ (\text{Ind}^G_{\tilde{G}}\lambda = \pi_1(\lambda) \oplus \pi_2(\lambda)) \ (\S 4.1)</td>
<td>\text{Ind}^G_{\tilde{G}}\lambda \ (\S 5.1)</td>
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<tr>
<td>\text{Ind}^G_{\tilde{G}}\sigma, \sigma \text{ cubic cuspidal} \ (\S 4.2)</td>
<td>\text{Ind}^G_{\tilde{G}}\tilde{\sigma} \ (\S 5.2)</td>
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<tr>
<td>{\pi^2(\lambda), \pi^<em>(\lambda)} \ (\pi^</em> \text{ described in Prop. 4.4}) \ (\S 4.1)</td>
<td>\text{Ind}^G_{\tilde{G}}\text{St}_{\tilde{H}}\left( (\lambda_1 \lambda_2) \cdot \left</td>
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<td>{\pi^<em>(\lambda), \pi^</em>(\lambda)^{<em>}} \ (\pi^</em> \text{ described in Prop. 4.4}) \ (\S 4.1)</td>
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<td>{ \text{Ind}^G_{\tilde{G}}\text{inf}<em>{\tilde{G}<em>v}\left( -R</em>{\tilde{C}}\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \right), \text{Ind}^G</em>{\tilde{G}}\text{inf}<em>{\tilde{G}<em>v}\left( -R</em>{\tilde{C}}\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \right), \text{Ind}^G</em>{\tilde{G}}\text{inf}<em>{\tilde{G}<em>v}\left( -R</em>{\tilde{C}}\varphi_2 \otimes \varphi_3 \otimes \varphi_1 \right), \text{Ind}^G</em>{\tilde{G}}\text{inf}_{\tilde{G}<em>v}\left( -R</em>{\tilde{C}}\varphi_3 \otimes \varphi_1 \otimes \varphi_2 \right), \ (\varphi_i \text{ distinct}) } \ (\text{Prop. 4.5})</td>
<td>\text{Ind}^G_{\tilde{G}<em>v}\text{inf}</em>{\tilde{G}<em>v}\left( -R</em>{\tilde{C}}\tilde{\varphi}_1 \otimes \tilde{\varphi}_2 \otimes \tilde{\varphi}_3 \right) \ (\text{Prop. 5.7})</td>
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restriction of scalars. Similarly, if $G$ is a $k_F$-group, $\overline{G}$ will denote $R_{k_E/k_F}(G)$. Whenever we use this notation, the extension $E/F$ will either be specified, or it will be understood from the context.

For every nonarchimedean local field $F$ and every reductive algebraic $F$-group $G$, one has an associated extended affine building $\mathcal{B}(G, F)$, as defined by Bruhat and Tits [3, 4]. As a $G$-set, $\mathcal{B}(G, F)$ is a direct product of an affine space (on which $G$ acts via translation) and the reduced building $\mathcal{B}^{\text{red}}(G, F)$, which depends only on $G/\mathbb{Z}$, where $\mathbb{Z}$ is the center of $G$. Note that $Z$ fixes $\mathcal{B}^{\text{red}}(G, F)$. For any extension $E/F$ of finite residue degree, $\mathcal{B}(G, F)$ always has a natural embedding into $\mathcal{B}(\overline{G}, F) = \mathcal{B}(G, E)$. To every point $x \in \mathcal{B}(G, F)$, there is an associated parahoric subgroup $G_x$ of $G$. The stabilizer of $x$ in $G$ contains $G_x$ with finite index. The pro-$p$-radical of $G_x$ is denoted $G_{x+}$, and the quotient $G_x/G_{x+}$ is the group of rational points of a connected reductive $k_F$-group $\overline{G}_x$. These objects depend only on the image of $x$ in $\mathcal{B}^{\text{red}}(G, F)$. Thus, in the case of a torus $T$, we may write $T_0$, $T_0+$, and $\overline{T}$ instead of $T_x$, $T_{x+}$, and $\overline{T}_x$, since these do not depend on the choice of $x$. More generally, $G_{0+}$ will denote the set of topologically unipotent elements in $G$.

We now present an elementary fact about the building that we will use several times throughout this paper.

**Lemma 2.1.** Let $\mathbb{Z}$ denote the center of $\overline{G}$, and let $y, z \in \mathcal{B}(\overline{G}, F)$ have distinct images in $\mathcal{B}^{\text{red}}(\overline{G}, F)$. Suppose $G_y$ is a maximal parahoric subgroup, $\gamma \in G_y$, and the image $\bar{\gamma}$ of $\gamma$ in $G_y$ is regular elliptic (i.e., $\bar{\gamma}$ belongs to no proper $k_F$-parabolic subgroup of $\overline{G}_y$). Then $\gamma \notin ZG_z$.

**Proof.** First, suppose $\gamma \in ZG_z \setminus G_z$. From [10, Lemma 4.2.1], $\gamma$ does not fix $z$. Therefore, $\gamma$ must act on some line containing $z$ via a nontrivial translation. By [10, Cor. 3.1.5], $\gamma$ cannot fix $y$, a contradiction.

Now suppose $\gamma \in G_z$. Then $\gamma \in G_x$ for all $x$ lying on the geodesic between $y$ and $z$. For such an $x$ that is close to but not equal to $y$, $G_x$ is a subgroup of $G_y$, and the image of $G_x$ in $G_y$ is the group of $k_F$-fixed points of a proper parabolic subgroup. Thus $\gamma \notin G_x$, a contradiction, and the lemma follows. \qed

If $G$ is a connected reductive group over a finite field, $\overline{T}$ is a maximal torus in $\overline{G}$, and $\theta$ is a (complex) character of $\overline{T}$, then let $R^\theta_G$ denote the corresponding Deligne-Lusztig virtual character of $G$ [12].

For any reductive algebraic group $G$ defined over a local or finite field, we have the following notation.

- $1_G$ will denote the trivial representation of $G$.
- $\text{St}_G$ will denote the Steinberg representation of $G$.
- For any character $\psi$ of $G$, $\text{St}_G(\psi)$ will denote $\text{St}_G \cdot \psi$.
- For any representation $\sigma$ of a subgroup $H$ of $G$, $\text{ind}_H^G \sigma$ will denote the representation of $G$ obtained from $\sigma$ via normalized compact induction.
- If $Z$ is the center of $\overline{G}$ and $\omega$ is a character of $Z$, then let $C(G, \omega)$ denote the space of complex-valued, locally constant functions $f$ on $G$ such that the support of $f$ is compact modulo $Z$, and $f(gz) = f(g)\omega(z)$ for all $g \in G$ and $z \in Z$.
- $G^{\text{reg}}$ denotes the set of regular semisimple elements of $G$.
- For any admissible, finite-length representation $\pi$ of $G$, let $\theta_\pi$ denote the character of $\pi$, considered either as a function on the set of elements or
conjugacy classes of $G$ (of $G_{\operatorname{reg}}$ in the local-field case), or as a distribution on an appropriate function space on $G$.

- Suppose $\varepsilon$ is an automorphism of $G$. Then $\varepsilon$ acts in a natural way on the set of equivalence classes of irreducible, admissible representations of $G$. Suppose $\pi$ is such a representation and $\pi \cong \pi'$. Let $\pi(\varepsilon)$ denote an intertwining operator from $\pi$ to $\pi'$. If $\varepsilon$ has order $\ell$, then we can and will normalize $\pi(\varepsilon)$ by requiring that the scalar $\pi(\varepsilon)^{\ell}$ equal 1. Then $\pi(\varepsilon)$ is well determined up to a scalar $\ell$th root of unity. The $\varepsilon$-twisted character of $\pi$ is the distribution $\theta_{\pi,\varepsilon}$ defined by $\theta_{\pi,\varepsilon}(f) = \operatorname{trace}(\pi(f)\pi(\varepsilon))$ for $f \in C_c^\infty(G)$. As with the character, the twisted character can be represented by a function (again denoted $\theta_{\pi,\varepsilon}$) on $G$ ($G_{\operatorname{reg}}$ in the local-field case). We may regard $\theta_{\pi,\varepsilon}$ as a function on the set of $\varepsilon$-twisted conjugacy classes.

  Note that $\theta_{\pi,\varepsilon}$ still makes sense when $\pi$ is an admissible, finite-length representation.

- For any maximal torus $T$ of $G$, let $W(T, G)$ denote the quotient of $T$ in its normalizer in $G$, and let $W_F(T, G)$ denote the group of $F$-points of the absolute Weyl group $N_G(T)/T$.

2.2. Shintani lifting. Suppose that $E/F$ is a finite, cyclic extension of local or finite fields, $\Gamma = \operatorname{Gal}(E/F)$, and $\tilde{G}$ is a connected reductive algebraic $F$-group. Let $\varepsilon$ denote a generator of $\Gamma$, and let $\ell$ denote the order of $\Gamma$. Then one can define a norm mapping from $G$ to $\tilde{G}$ by

\[ x \mapsto x \cdot \varepsilon(x) \cdot \cdots \cdot \varepsilon^{\ell-1}(x). \]

If $x$ is defined over $F$ then, in general, the most that one can say about the image of $x$ is that its conjugacy class in $\tilde{G}$ is defined over $F$. If $F$ is local and $\tilde{G}$ has a simply connected derived group, then such a conjugacy class must have $F$-points [23]. Thus, an $F$-point $x \in \tilde{G}$ determines a stable conjugacy class in $G$. Any stable, $\varepsilon$-twisted conjugate of $x$ determines the same stable conjugacy class in $G$. Thus, we have a map $N_{E/F}^{\varepsilon}$ from the set of stable, $\varepsilon$-twisted conjugacy classes of $\tilde{G}$ to the set of stable conjugacy classes in $G$. If $x$ commutes with its Galois conjugates, then we may and will define $N_{E/F}^{\varepsilon}(x) \in G$ via the formula above.

Call $g \in \tilde{G}$ $\varepsilon$-regular if $N(g)$ is regular. Let $\tilde{G}_{\varepsilon-\operatorname{reg}}$ denote the set of $\varepsilon$-regular elements.

If $\Pi$ and $\tilde{\Pi}$ are finite sets of representations of $G$ and $\tilde{G}$, respectively, we say that $\tilde{\Pi}$ is the Shintani lift (or base change) of $\Pi$ if

\[ \Theta_{\tilde{\Pi},\varepsilon}(g) = \Theta_{\Pi}(N(g)) \]

for all $g \in \tilde{G}$ (all $g \in \tilde{G}_{\varepsilon-\operatorname{reg}}$ in the local-field case), where $\Theta_{\Pi}$ and $\Theta_{\tilde{\Pi},\varepsilon}$ are non-trivial stable (resp. $\varepsilon$-stable) linear combinations of the characters (resp. $\varepsilon$-twisted characters) of the elements of $\Pi$ (resp. $\tilde{\Pi}$).

If $T$ is an $F$-torus then for any character $\lambda$ of $T$, define the character $\tilde{\lambda}$ of $\tilde{T}$ by $\tilde{\lambda} = \lambda \circ N_{E/F}^{\varepsilon}$.

2.3. Notation related to unitary groups. From now on, fix a nonarchimedean local field $F$ of characteristic zero with finite residue field $k_F$ of odd order $q$. Let $E$ be the unramified quadratic extension of $F$. Let $E^1$ (resp. $k_E^1$) denote the kernel
of the norm from $E$ to $F$ (resp. $k_E$ to $k_F$). Let $\varepsilon$ denote the nontrivial element of the Galois group $\Gamma = \text{Gal}(E/F)$.

Let $\tilde{G}$ denote a unitary group in three variables defined with respect to $E/F$. Then $\tilde{G}$ is uniquely determined up to isomorphism, and we can and will assume that $\tilde{G}$ is the unitary group defined by the Hermitian matrix

$$
\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Then

$$
G = \tilde{G}(F) = \{ g \in \text{GL}(3, E) : g \Phi = \Phi \}
$$

and $\tilde{G} = \text{GL}(3, E)$. Let $G$ denote the corresponding algebraic group over $k_F$. Then $G = \text{GL}(3, k_F)$.

Let $Z$ denote the center of $G$. So, following our notational conventions, $\tilde{Z}$ is the center of $\tilde{G}$.

Let $\mathcal{B} = \mathcal{B}(\tilde{G}, F)$ and $\tilde{\mathcal{B}} = \mathcal{B}(\tilde{G}, F) = \mathcal{B}(G, E)$. Note that $\varepsilon$ acts on $\tilde{\mathcal{B}}$, and we may and will identify the set of fixed points $\tilde{\mathcal{B}}^\varepsilon$ with $\mathcal{B}$.

Since $\tilde{G}$ is $F$-quasisplit, it contains $F$-Borel subgroups. In particular, $\tilde{\mathcal{B}}$ must contain some $\varepsilon$-invariant apartment $\tilde{\mathcal{A}}$ with more than one $\varepsilon$-fixed point. Choose an $\varepsilon$-fixed point $y$ in an $\varepsilon$-invariant minimal facet in $\tilde{\mathcal{A}}$, and an $\varepsilon$-invariant alcove $\tilde{\mathcal{F}}$ in $\tilde{\mathcal{A}}$, such that the closure of $\tilde{\mathcal{F}}$ contains $y$. (Let $\mathcal{F}$ denote the set of $\varepsilon$-fixed points of $\tilde{\mathcal{F}}$.) Then these choices determine an $F$-Borel subgroup $\mathcal{B}$ together with a Levi factor $M$ of $\mathcal{B}$. Note that $M$ is isomorphic to $E^\times \times E^1$. We may assume that our choices of $y$ and $\tilde{\mathcal{F}}$ allow us to realize $\mathcal{B}$ explicitly as the group of upper triangular matrices in $G$, and $M$ as the group of diagonal matrices.

The boundary of $\mathcal{F}$ contains two points: the previously chosen point $y$, and another point that we will denote $z$. Note that $\hat{\mathcal{F}}$ is the direct product of a one-dimensional affine space and an $\varepsilon$-invariant equilateral triangle $\Delta$ in the reduced building of $\tilde{G}$ (which we will identify with a subset of $\tilde{\mathcal{B}}$), $y$ the $\varepsilon$-fixed vertex of $\Delta$, and $z$ is the midpoint of the wall of $\Delta$ that is opposite $y$. In $\tilde{\mathcal{B}}$, $y$ and $z$ are both vertices, but only $y$ is hyperspecial.

Consider the map $\lambda$: $U(1) \to \tilde{G}$ given by $t \mapsto \text{diag}(1, t, 1)$. Since $\tilde{U}(1) \cong \text{GL}(1)$, we actually have a one-parameter subgroup of $\tilde{G}$. In the usual way, $\lambda$ determines a parabolic $F$-subgroup $\tilde{P} = \tilde{P}_\lambda$ of $\tilde{G}$, together with a Levi decomposition of $\tilde{P}$. Let $\tilde{H}$ denote the corresponding Levi factor. Then $\tilde{H}$ is the group of invertible matrices of the form

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & * \end{pmatrix}.
$$

This subgroup arises via restriction of scalars from a subgroup $H$ of $G$. Note that $\mathcal{H}$ is an $E$-Levi, but not $F$-Levi, subgroup of $G$. It is an endoscopic group for $G$, isomorphic to $U(1, 1) \times U(1)$.

Similarly, we can define a subgroup $H$ of $G$ and a parabolic $k_F$-subgroup $\tilde{P}$ of $\tilde{G}$ with Levi factor $\tilde{H}$. Note that $\tilde{G} \cong G$ and $\tilde{G} \cong H$.

Up to conjugacy, $\tilde{H}$ contains two $F$-tori that are isomorphic to $U(1) \times U(1) \times U(1)$. The group of $F$-points of one of these tori fixes a hyperspecial vertex, and the group of $F$-points of the other fixes a non-hyperspecial vertex. Pick such a torus whose $F$-points fix $y$ (resp. $z$) and call it $\tilde{C}$ (resp. $\tilde{C}'$). Given the right choices, we can
and will realize $C$ as the set of matrices of the form
\[
\gamma = \begin{pmatrix}
\frac{\gamma_1}{2} & 0 & \frac{\gamma_1 - 1}{2} \\
0 & \gamma_2 & 0 \\
\frac{\gamma_1 - 1}{2} & 0 & \frac{\gamma_1}{2}
\end{pmatrix}
\]
where $\gamma_i \in U(1)$. We define the torus $C \subset G$ similarly. We identify $C$ (and similarly $\overline{C}$) with $U(1)^3$ via the map $\gamma \mapsto (\gamma_1, \gamma_2, \gamma_3)$. We will realize $C'$ as $\nu C \nu^{-1}$, where
\[
\nu = \begin{pmatrix}
\frac{1}{\sqrt{\omega_F}} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \sqrt{\omega_F}
\end{pmatrix}.
\]
(The ambiguity in the choice of square root of $\omega_F$ has no effect.)

Let $B_y$ (resp. $B_z$) denote the Borel subgroup of $G_y$ (resp. $G_z$) determined by $F_y$ (resp. $F_z$). For any $E$-group $L$, let $N_{\ell_E/F} = N_{E/F}$. When $L = G$, we simply write $N$. Similarly, for any $k_F$-group $L$, let $N_{k_{E/F}} = N_{k_{E/K}}$. When $L = G$, we simply write $N$.

For any subgroup $S \subset \widetilde{G}$, let $\det_S$ denote the restriction of the determinant to $S$. We will omit the subscript when it is clear from the context. Similar notation holds for subgroups of $\widetilde{G}$.

2.4. Cartan subgroups of $G$. For a quadratic extension $L/K$, denote by $U(1, L/K)$ the unitary group in one variable over $K$ defined with respect to $L/K$. Up to stable conjugacy, there are four kinds of Cartan subgroup of $G$. In the notation of [23], they are isomorphic to:

(2.4–0) $R_{E/F}(GL(1)) \times U(1, E/F)$,
(2.4–1) $U(1, E/F) \times U(1, E/F) \times U(1, E/F)$,
(2.4–2) $R_{E/F}(U(1, EK/K)) \times U(1, E/F)$ for $K$ a ramified quadratic extension of $F$,
(2.4–3) $R_{L/F}(U(1, EL/L))$ for $L$ a cubic extension of $F$.

2.5. Representation theory of $G$, $H$, and $C$. A reference for much of this section is [27].

Representations of $G$. Let $B$ denote a Borel subgroup of $G$ with Levi factor $M$, and let $\theta$ be a character of $M$. Then the induced representation $\text{ind}_{B}^{G}\theta$ is irreducible except when $\theta$ extends to a character of $H$. In this case, the induced representation is a sum of two irreducible components. If $\theta$ extends to a character $\theta_0$ of $G$, then these components are $\theta_0$ and $\text{St}_{G}(\theta_0)$.

Let $L$ denote a cubic unramified extension of $E$. Then $G$ contains a torus $S$ that is isomorphic to the kernel of the norm map from $k_{EL}$ to $k_L$. Let $T$ be either $S$ or $C$. For any character $\theta$ of $T$ with trivial stabilizer in $W_{k_L}(T, G)$, we have a Deligne-Lusztig cuspidal representation whose character is $-R_{L/F}^{G}\theta$. For $T = S$, we will call such representations “cubic cuspidal representations.”

The other irreducible representations of $G$ have the form $\tau \cdot \psi$, where $\tau$ is the cuspidal unipotent representation and $\psi$ is a character.
Representations of $H$. As above let $B$ be a Borel subgroup of $G$ with Levi factor $M$ and let $\theta$ be a character of $M$. The induced representation $\text{ind}_{B \cap H}^H \theta$ is irreducible except when $\theta$ extends to a character $\theta_1$ of $H$. In this case, the induced representation is the sum of $\theta_1$ and $\text{St}_H(\theta_1)$.

The remaining representations of $H$ are the Deligne-Lusztig cuspidal representations, whose characters are of the form $-R_G^C \theta$ for $\theta \in \text{Hom}(C, C^\times)$ in general position with respect to the action of $W_{k_F}(C, H)$.

Representations of $C$. We will need a technical result on linear combinations of characters of $C$. Let $A$ be a finite abelian group of order $n$, and let $\chi_1, \ldots, \chi_n$ be the irreducible characters of $A$. The following three lemmas concern characters of products of copies of $A$.

Lemma 2.2. Let
\[ f = \sum_{i=1}^n a_i \chi_i, \]
where $a_i \in \mathbb{C}$. Suppose that $f$ vanishes off of a subset of $A$ of size 2. Then either $f = 0$, or the number of $i$ such that $a_i \neq 0$ is at least $n/2$.

Proof. Let $\{a, b\}$ be the above subset of $A$. We have
\[ n a_i = n \cdot \langle f, \chi_i \rangle = f(a) \bar{\chi}_i(a) + f(b) \bar{\chi}_i(b). \]
Assume $f \neq 0$. If $f(b) = 0$, then for all $i$, $a_i = f(a) \bar{\chi}_i(a)/n \neq 0$. If $f(b) \neq 0$, then $a_i \neq 0$ unless $\bar{\chi}_i(ba^{-1}) = -f(a)/f(b)$. Since $ba^{-1} \neq 1$, this equality holds for at most $n/2$ values of $i$. \qed

Lemma 2.3. Let $N$ be the subset of $A \times A$ consisting of all elements $(a, b)$ such that $a \neq b$. Suppose that for some $a_{ij} \in \mathbb{C}$,
\[ f = \sum_{i,j} a_{ij} \chi_i \otimes \chi_j \]
vanishes on $N$. Then either $f = 0$ or at least $n$ of the $a_{ij}$ are nonzero.

Proof. Assume $f \neq 0$. Fix $a \in A$. Evaluating $f$ at $(a, b)$ for $b \neq a$, we obtain that the function
\[ \sum_j \left( \sum_i a_{ij} \chi_i(a) \right) \chi_j \]
on $A$ vanishes on $A - \{a\}$. It follows easily that either this function vanishes on $A$, or for all $j$, the coefficient $\sum_i a_{ij} \chi_i(a)$ is nonzero. The former case cannot happen since $f \neq 0$. In the latter case, it follows that for all $j$, at least one coefficient $a_{ij}$ must be nonzero. Hence at least $n$ of the $a_{ij}$ must be nonzero. \qed

Lemma 2.4. Let $N'$ be the subset of $A \times A \times A$ consisting of all elements $(a, b, c)$ such that $a, b, c$ are distinct. Suppose that for some $a_{ijk} \in \mathbb{C}$,
\[ f = \sum_{i,j,k} a_{ijk} \chi_i \otimes \chi_j \otimes \chi_k \]
vanishes on $N'$. Then either $f$ vanishes on $A \times A \times A$ or at least $n/2$ of the $a_{ijk}$ are nonzero.
Proof. Assume $f \neq 0$. Fix $a \neq b$ in $A$. Then the function
\[ \sum_k \left( \sum_{i,j} a_{ijk} \chi_i(a) \chi_j(b) \right) \chi_k \]
on $A$ vanishes off of $\{a, b\}$. Hence, by Lemma 2.2, either this function vanishes on $A$, or for at least $n/2$ values of $k$, the coefficient $\sum_{i,j} a_{ijk} \chi_i(a) \chi_j(b)$ is nonzero. In the latter case, for each such $k$, at least one coefficient $a_{ijk}$ must be nonzero. Hence at least $n/2$ of the $a_{ijk}$ are nonzero.

We may therefore assume that the former case holds for all pairs $a \neq b$. By the linear independence of characters, the coefficient $\sum_{i,j} a_{ijk} \chi_i(a) \chi_j(b)$ must vanish for all $k$ and all pairs $a \neq b$. Since $f \neq 0$, $a_{i',j',k'} \neq 0$ for some $i', j', k'$. Thus the function $\sum_{i,j} a_{ijk} \chi_i \otimes \chi_j$ on $A \times A$ vanishes on the set $N$ of Lemma 2.3, but it does not vanish on $A \times A$ since $a_{i',j',k'} \neq 0$. Hence Lemma 2.3 implies that at least $n$ of the coefficients $a_{ijk}$ must be nonzero. 

Corollary 2.5. Suppose that
\[ \sum_{\chi \in \text{Hom}(C, C^\times)} a_{\chi} \chi \]
vanishes on $C \cap G^{\text{reg}}$, where $a_{\chi} \in C$. Then either this linear combination vanishes on $C$ or at least $(q + 1)/2$ of the $a_{\chi}$ are nonzero. 

Corollary 2.6. Suppose that $q > 59$, and let $f = \sum a_{\chi} \chi$ be a linear combination of at most 30 characters of $C$. If $f$ vanishes on $C \cap G^{\text{reg}}$, then $f$ vanishes on $C$. 

2.6. Shintani lifting for $G$ and $H$. According to [27], the irreducible characters of $G$ are of the form $\pm R^G_{\tilde{\tau}} \theta$, where $L$ is the connected centralizer of some semisimple element of $G$, and $\theta$ is the twist of a unipotent character of $L$ by a one-dimensional character in general position. Moreover, one obtains a cuspidal character of $G$ precisely when $L$ is an elliptic torus or when $L = G$ and $\theta$ is a twist of the unique cuspidal unipotent character of $G$.

By [17], our assumption that $k_F$ has odd characteristic guarantees the existence of Shintani descent from $	ilde{G}$ to $G$. In [14], Digne gives a general proof that Shintani descent is compatible with Deligne-Lusztig induction. In particular, if $\sigma$ is an irreducible representation of $G$ with character $\pm R^G_{\tilde{\tau}} \theta$ ($\theta$ a character of $L$), then the character of the Shintani lift $\tilde{\sigma}$ of $\sigma$ from $G$ to $\tilde{G}$ is of the form $\pm R^\tilde{G}_{\tilde{\tau}} \tilde{\theta}$, where $\tilde{\theta}$ is the Shintani lift of $\theta$. Now $L$ is a Levi factor of a parabolic subgroup of $\tilde{G}$ unless $L$ is isomorphic to the torus $S$ defined in §2.5. Hence $\tilde{\sigma}$ is a parabolically induced representation unless $L \cong S$ or $L = G$. In the former case, $S$ is an elliptic torus isomorphic to $kL^* L$ and $\tilde{\sigma}$ is cuspidal. In the latter case, $\sigma$ is a one-dimensional representation $\varphi \circ \det_{\tilde{G}}$, a twist $\text{St}_G(\varphi \circ \det_{\tilde{G}})$ of the Steinberg representation, or a twist $\tau(\varphi \circ \det_{\tilde{G}})$ of the cuspidal unipotent representation. One shows easily that the Shintani lift $\tilde{\sigma}$ is, respectively, $\tilde{\varphi} \circ \det_{\tilde{G}}$, $\text{St}_G(\tilde{\varphi} \circ \det_{\tilde{G}})$, or $\tilde{\tau}(\tilde{\varphi} \circ \det_{\tilde{G}})$, where $\tilde{\tau}$ is the unipotent representation of $\tilde{G}$ not equivalent to $1_{\tilde{G}}$ or $\text{St}_G$. The remaining representations of $G$ are those whose characters are of the form $R^G_{\tilde{\tau}} \theta$. By [14], the Shintani lifts of such representations are representations induced from $P$. Hence the cubic cuspidal representations of $G$ are exactly those irreducible representations of $G$ whose Shintani lifts are cuspidal.
We now consider Shintani lifting for irreducible representations of $H$. From §2.5, most such representations have characters of the form $±R^H_x\theta$. From Digne [14], the Shintani lift of such a representation has character $±R^H_{\tilde{\theta}}$.

The remaining representations of $H$ are the one-dimensional representations $\varphi \circ \det_{\underline{\mathbb{H}}}$ and the Steinberg representations $\text{St}_H(\varphi \circ \det_{\underline{\mathbb{H}}} \circ \psi)$. It is easy to see that the respective Shintani lifts of these representations are $\varphi \circ \det_{\underline{\mathbb{H}}}$ and $\text{St}_H(\varphi \circ \det_{\underline{\mathbb{H}}})$.

2.7. Depth-zero representations of $G$.

Principal series of $G$. For $\lambda \in \text{Hom}(M, \mathbb{C}^\times)$, there exist unique characters $\lambda_1 \in \text{Hom}(E^\times, \mathbb{C}^\times)$ and $\lambda_2 \in \text{Hom}(E^1, \mathbb{C}^\times)$ such that

$$(2.7.1)\quad \lambda \left( \begin{array}{ccc} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1} \end{array} \right) = \lambda_1(\alpha)\lambda_2(\alpha^{-1}\beta),$$

where $\alpha \in E^\times$, $\beta \in E^1$. By [18], $\text{ind}_E^G \lambda$ is irreducible except for in the following cases:

- (2.7PS–1) $\lambda_1 = | \cdot |_{E}^{\pm1}$
- (2.7PS–2) $\lambda_1|_{F^\times} = \omega_{E/F} | \cdot |_{F}^{\pm1}$
- (2.7PS–3) $\lambda_1$ is nontrivial and $\lambda_1|_{F^\times}$ is trivial.

In case (2.7PS–1), $\text{ind}_E^G \lambda$ has two constituents: the one-dimensional representation $\psi = \lambda_2 \circ \det$, and the square-integrable Steinberg representation $\text{St}_{G}(\psi)$.

In case (2.7PS–2), $\text{ind}_E^G \lambda$ also has two constituents: a square-integrable representation $\pi_2(\lambda)$ and a non-tempered unitary representation $\pi_1(\lambda)$.

In case (2.7PS–3), $\text{ind}_E^G \lambda$ decomposes into a direct sum $\pi_1(\lambda) \oplus \pi_2(\lambda)$.

By [21], $\text{ind}_E^G \lambda$ has depth zero if and only if $\lambda$ has depth zero.

Other representations of $G$. Since $G$ has no non-minimal proper parabolic subgroups, the remaining irreducible representations are all supercuspidal. From either [21] or [20], we know that all such representations have a unique expression of the form $\text{ind}_{G_x}^G \sigma$, where $x = y$ or $z$, and $\sigma$ is the inflation to $G_x$ of an irreducible cuspidal representation $\overline{\sigma}$ of $G_x$. The representations $\overline{\sigma}$ are classified in §2.5. Based on this classification, we have the following kinds of supercuspidal representation of depth zero.

- (2.7SC–1) $\text{ind}_{G_x}^G \sigma$, where $\overline{\sigma}$ is a cubic cuspidal representation of $G_y \cong G$.
- (2.7SC–2) $\text{ind}_{G_x}^G \sigma$, where $\overline{\sigma}$ is a cuspidal representation of $G_y$ with character $-R_{C_0}^{G_y} \varphi$ and $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ is a regular character of $C$ (with respect to $W_{k,F}(C_0, G_y)$).
- (2.7SC–3) $\text{ind}_{G_x}^G \sigma$, where $\overline{\sigma}$ is the twist $\tau \cdot (\eta \circ \det)$ of the cuspidal unipotent representation $\tau$ of $G_y$, and $\eta \in \text{Hom}(k_{E}^{1}, \mathbb{C}^\times)$.
- (2.7SC–4) $\text{ind}_{G_x}^G \sigma$, where $\overline{\sigma}$ is a cuspidal representation of $G_z \cong H$ with character $-R_{C_0}^{G_z} \varphi$ and $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ is a regular character of $C'$ (with respect to $W_{k,F}(C_0, G_z)$). Recall that, according to our notational conventions, $C'$ is the finite reductive quotient of the (unique) parahoric subgroup of $C$. 

3. Description of depth-zero $L$-packets and explicit base change for
unitary groups in two variables

In this section we give brief descriptions of the depth-zero $L$-packets for
the quasi-split group $U(1,1)(F)$ and the compact group $U(2)(F)$, as well as their base
change lifts to $GL_2(E)$. We omit the proofs as they are entirely analogous to (but
less complicated than) those for $U(2,1)$. Let $H^0$ be the group $U(1,1)$, which we
will view as the subgroup of $H$ consisting of all matrices of the form
\[
\begin{pmatrix}
0 & a \\
0 & 1
\end{pmatrix}.
\]
For every subgroup $L$ of $G$, let $L^0$ denote the subgroup $L \cap H^0$ of $H^0$. Let $H^1$
do denote the compact inner form $U(2)$ of $H^0$. Note that $H^1(F) \cong GL_2(E)$.

From our descriptions, it will be clear that the analogue of Theorem 1.1 holds
for unitary groups in two variables.

3.1. Depth-zero $L$-packets for $U(1,1)$. The $L$-packets of $H^0$ are the $PGL_2(F)$-
orbits on the set of equivalence classes of irreducible admissible representations of
$H^0$ [23, §11.1]. We first describe the principal series $L$-packets.

Let $\lambda \in \text{Hom}(M^0, \mathbb{C}^\times) = \text{Hom}(E^\times, \mathbb{C}^\times)$. According to [23, §11.1], the principal
series $\text{ind}^{H_0}_{B_0} \lambda$ is irreducible except in the cases
\begin{enumerate}
\item $\lambda|_{B^\times} = | \cdot |_F^{\frac{3}{2}}$;
\item $\lambda|_{B^\times} = \omega_{E/F}$.
\end{enumerate}

In the first case, $\text{ind}^{H_0}_{B_0} \lambda$ has two constituents: the one-dimensional representation
$\psi = \mu \circ \det$, where $\mu \circ \mathcal{N} = \lambda| |_E^{\frac{1}{2}}$, and the Steinberg representation $\text{St}_G(\psi)$. In
the second case, $\text{ind}^{H_0}_{B_0} \lambda$ decomposes into a direct sum $\pi_1(\lambda) \oplus \pi_2(\lambda)$ of irreducible
representations. By [21], $\text{ind}^{H_0}_{B_0} \lambda$ has depth zero if and only if $\lambda$ has depth zero.

The principal series $L$-packets of $G$ are as follows [23, §11.1]. (Here $\lambda$ and $\psi$
denote one-dimensional representations of $M^0$ and $H^0$, respectively.)

\begin{enumerate}
\item $\{\text{ind}^{H_0}_{B_0} \lambda\}$, where $\text{ind}^{H_0}_{B_0} \lambda$ is irreducible;
\item $\{\psi\}$;
\item $\{\text{St}_{H_0}(\psi)\}$;
\item $\{\pi_1(\lambda), \pi_2(\lambda)\}$, where $\text{ind}^{H_0}_{B_0} \lambda$ is reducible of the second type described
above.
\end{enumerate}

The remaining irreducible representations and $L$-packets of $G$ are all supercus-
pidal. The depth zero supercuspidals of $H^0$ have a unique expression of the form
$\text{ind}^{H_0}_{B_0} \sigma$, where $v = y$ or $z$, and $\sigma$ is the inflation to $H^0_0$ of an irreducible cuspidal
representation $\sigma$ of $H^0_v$. Let $C^0_v$ be $C \cap H^0$ if $v = y$, and $C' \cap H^0$ if $v = z$. Then
the character of such a representation $\sigma$ must be of the form $-R^{C^0_v}_{C^0_v} \varphi$, where $\varphi$
is a character of $C^0_v$ in general position. Since $C^0_v \cong k_E^1 \times k_E^1$, we may view any such
character as having the form $\varphi_1 \otimes \varphi_2$, where the $\varphi_i$ are distinct characters of $k_E^1$.

Fix a cuspidal representation $\sigma$ of $H^0_v$. Viewing it as a representation of $H^0_v$, we
inflate it to a representation $\sigma_v$ of $H^0_v$. Let $\pi_v = \text{ind}^{H_0}_{B_0} \sigma_v$. Then $\{\pi_y, \pi_z\}$ is a
depth-zero supercuspidal $L$-packet of $H^0$. Conversely, all such $L$-packets are of this
form. If $\pi_y$ and $\pi_z$ are formed from the character $\varphi_1 \otimes \varphi_2$ of $C^0$ as above, then for
future reference call this $L$-packet $\Pi^0_{\varphi_1, \varphi_2}$. 

3.2. Base change lifts for $U(1, 1)$. By \cite[§11.4]{11.4}, the base change lifts of principal series $L$-packets of $H^0$ are as follows. Let $\lambda \in \text{Hom}(M^0, \mathbb{C}^\times)$.

(i) If $\text{ind}_{B^0} H^0 \lambda$ is irreducible and $\text{ind}_{B^0} H^0 \tilde{\lambda}$ is irreducible, then the base change lift of $\text{ind}_{B^0} H^0 \lambda$ is $\text{ind}_{B^0} H^0 \tilde{\lambda}$.

(ii) If $\text{ind}_{B^0} H^0 \lambda$ is irreducible but $\text{ind}_{B^0} H^0 \tilde{\lambda}$ is reducible, then $\lambda|_{E^\times} = |\frac{1}{E} \omega_{E/F}|$, and the base change lift of $\text{ind}_{B^0} H^0 \lambda$ is $\lambda|_{E^\times} \circ \det$.

(iii) If $\lambda|_{E^\times} = |\frac{1}{E} \cdot F|$, let $\psi$ be the one-dimensional representation $\mu \circ \det_{H^0}$, where $\mu \circ N = \lambda|_{E^\times}$. Then the lift of the $L$-packet consisting of the constituent $\psi$ (resp., the Steinberg constituent $\text{St}_{H^0}(\psi)$) of $\text{ind}_{B^0} H^0 \lambda$ is the one-dimensional constituent $\tilde{\psi} = (\lambda|_{E^\times} \circ \det_{H^0})$ (resp., the Steinberg constituent $\text{St}_{H^0}(\tilde{\psi})$) of $\text{ind}_{B^0} H^0 \tilde{\lambda}$.

(iv) If $\lambda|_{E^\times} = \omega_{E/F}$, then the lift of the $L$-packet $\{\pi_1(\lambda), \pi_2(\lambda)\}$ is $\text{ind}_{B^0} H^0 \tilde{\lambda}$.

The base change lift of the depth-zero supercuspidal $L$-packet $\Pi^0_{\varphi_1, \varphi_2}$ is the principal series representation $\text{ind}_{B^0} H^0 \varphi^*$, where $\varphi^*$ is the character $\varphi_i \omega_{E/E} \otimes \hat{\varphi}_i \omega_{E/E}$ of $E^\times \times E^\times \cong \hat{M}^0$. Here, $E^\times$ is an unramified quadratic extension of $E$, and $\hat{\varphi}_i$ is the inflation to $E^\times$ of the character $\varphi_i$ of $k_E^\times$.

3.3. Depth-zero $L$-packets for $U(2)$. Since $H^1$ is compact, it has only one parahoric subgroup (and in fact is equal to it). The finite reductive quotient $H^1$ is isomorphic to $k_E^1 \times k_E^1$. Thus, every irreducible, depth-zero representation of $H^1$ has the form $\text{infl}(\varphi_1 \otimes \varphi_2)$, the inflation to $H^1$ of a character of $H^1$.

Let

$$\Pi^1_{\varphi_1, \varphi_2} = \begin{cases} \{\text{infl}(\varphi_1 \otimes \varphi_2), \text{infl}(\varphi_2 \otimes \varphi_1)\} & \text{if } \varphi_1 \neq \varphi_2, \\ \{\text{infl}(\varphi_1 \otimes \varphi_2)\} & \text{if } \varphi_1 = \varphi_2. \end{cases}$$

Then we declare the $\Pi^1_{\varphi_1, \varphi_2}$ to be the $L$-packets for $H^1$. These $L$-packets are chosen so as to make the correspondence $\text{JL}$ given in §3.4 work properly.

3.4. Base change lifts for $U(2)$ via a Jacquet-Langlands-like correspondence. Since $H^1$ is an inner form of $H^0$, we can obtain a base change lift if we can associate each $L$-packet for $H^1$ to one for $H^0$. This association will be similar to the Jacquet-Langlands correspondence (or “Abstract Matching Theorem” \cite{11.22, 22}). That is, given an $L$-packet $\Pi^1$ for $H^1$, we want to find an $L$-packet $\Pi^0$ for $H^0$ such that

$$(3.4.1) \quad \sum_{\pi \in \Pi^1} \theta_{\pi}(g_1) = \pm \sum_{\pi \in \Pi^0} \theta_{\pi}(g_0)$$

for all regular $g_1 \in H^1$ and $g_0 \in H^0$ whose stable conjugacy classes are associated in a natural way.

Define a map $\text{JL}$ from the depth-zero $L$-packets of $H^1$ to those of $H^0$ by

$$\text{JL}(\Pi^1_{\varphi_1, \varphi_2}) = \Pi^0_{\varphi_1, \varphi_2}$$

if $\varphi_1 \neq \varphi_2$. If $\varphi = \varphi_1 = \varphi_2$, then we define $\text{JL}(\Pi^1_{\varphi_1, \varphi_2})$ as follows. Form the character $\varphi \circ N$ of $k_E^1$, which we can then inflate to a character $\lambda$ of $E^\times$. Now let $\text{JL}(\Pi^1_{\varphi_1, \varphi_2})$ be the Steinberg component of $\text{ind}_{B^0} H^0 \varphi \cdot \det$. More specifically, this representation is $\text{St}_{H^0}(\mu \circ \det)$, where $\mu \circ N = \lambda|_{E^\times}$, as in §3.1.
It is not difficult to see that $JL$ is the only correspondence that satisfies (3.4.1) for all $g_1 \in H^1$ whose image in $H^1$ is regular. Thus, if we assume that there is a Jacquet-Langlands-like correspondence from the depth-zero $L$-packets of $H^0$ to those of $H^1$, then it must be $JL$.

4. Description of depth-zero $L$-packets and $A$-packets for $G$

In almost all cases, $L$-packets and $A$-packets are the same. In one case (see below), a certain principal series $L$-packet is enlarged to form an $A$-packet. Thus, while the $L$-packets constitute a partition of the set of equivalence classes of irreducible representations, the $A$-packets do not.

4.1. $L$-packets consisting of principal series constituents. The following proposition is due to Rogawski [23, §12.2].

**Proposition 4.1.** The $L$-packets of $G$ that consist entirely of principal series constituents all have one of the following forms (where $\lambda$ and $\psi$ denote one-dimensional representations of $M$ and $G$, respectively):

1. $(4.1–1) \{\text{ind}^L_{P} \lambda\}$, where $\text{ind}^L_{G} \lambda$ is irreducible;
2. $(4.1–2) \{\psi\}$;
3. $(4.1–3) \{\text{St}^G_{G} (\psi)\}$, where $\text{ind}^L_{G} \lambda$ is reducible of type (2.7PS–3);
4. $(4.1–4) \{\pi_1(\lambda), \pi_2(\lambda)\}$, where $\text{ind}^L_{G} \lambda$ is reducible of type (2.7PS–2).

In the last case, $\pi_n(\lambda)$ is contained in the $A$-packet $\Pi(\lambda) = \{\pi_n(\lambda), \pi_s(\lambda)\}$, where $\pi_s(\lambda)$ is the supercuspidal representation that sits inside an $L$-packet with the square-integrable principal series constituent $\pi_2(\lambda)$. In the depth-zero setting, the representation $\pi_s(\lambda)$ will be explicitly described in §4.3.

4.2. Singleton supercuspidal $L$-packets. In this section, we characterize the stable supercuspidal representations of $G$ of depth zero in terms of inducing data.

**Proposition 4.2.** A supercuspidal representation $\pi$ of $G$ of depth zero is stable if and only if $\pi$ is of the form $\text{ind}^G_{G_y} \sigma$, where $\sigma$ is the inflation to $G_y$ of a cubic cuspidal representation $\overline{\sigma}$ of $G_y$.

**Proof.** Let $\pi$ be a representation of the above form. Let $\gamma$ be an element of $G^\text{reg}$ and let $\gamma'$ be a stable conjugate of $\gamma$. We will show that

\[(4.2.1) \theta_\pi(\gamma) = \theta_\pi(\gamma').\]

The conjugacy classes contained within the stable conjugacy class of $\gamma$ are parametrized by

\[\text{Ker}\{H^1(F, G_y) \to H^1(F, G)\}\]

(see [23, §3.1]), where $G_y$ is the centralizer of $\gamma$ in $G$. If $\gamma$ is contained in a Cartan subgroup of $G$ of type (2.4–0) or (2.4–3), then this kernel is trivial by [23, §3.6] so any stable conjugate $\gamma'$ of $\gamma$ is a conjugate of $\gamma$. Hence $\theta_\pi(\gamma) = \theta_\pi(\gamma')$. Therefore, we may assume that $\gamma$ is contained in a Cartan subgroup $T$ of type (2.4–1) or (2.4–2).

For any regular, depth-zero $X$ in the dual of the Lie algebra of a cubic torus in $G$, the germ $\theta_\pi|_{G_{13}}$ coincides with a constant multiple of the Fourier transform of the orbital integral corresponding to $X$. (This follows from Corollaire III.10 and Proposition III.8 of [29]. It also follows from the proof of the main theorem of [1].)
The Weyl group of a cubic torus acts via the Galois group, so two regular elements of the torus are conjugate if and only if they are stably conjugate. Moreover, every stable conjugate of a cubic torus is conjugate to it. Therefore, this orbital integral is stable. From [28], the Fourier transform of a stable distribution is stable. Thus, \( \theta_\pi|_{G_{0+}} \) is stable. If \( z \in Z \), it is clear that \( \theta_\pi(\gamma z) = \theta_\pi(\gamma')z \) if and only if \( \theta_\pi(\gamma) = \theta_\pi(\gamma') \). Thus, \( \theta_\pi|_{ZG_{0+}} \) is stable.

It follows that (4.2.1) holds if \( \gamma \in ZT_{0+} \). Therefore, suppose that \( \gamma \notin ZT_{0+} \). We will show that \( \theta_\pi \) vanishes at all stable conjugates of \( \gamma \) (including \( \gamma \) itself), thus establishing (4.2.1). Let \( \gamma'' \) be a stable conjugate of \( \gamma \). If no conjugate of \( \gamma'' \) is contained in \( G_y \), then \( \theta_\pi(\gamma'') = 0 \) from Proposition 7.1. So assume \( \gamma'' \in G_y \). It follows easily from our assumptions on \( \gamma \) that the characteristic polynomial of the image \( \gamma'' \) of \( \gamma'' \) in \( G_y \) is reducible over \( k_E \) and that its roots are not all the same.

But then the semisimple part of \( \gamma'' \) is not contained in a cubic torus of \( G_y \) so, by [27, 6.9], it follows that \( \theta_\pi(\gamma'') = 0 \). Thus \( \theta_\pi(\gamma'') = 0 \) by Proposition 7.1.

Conversely, suppose that \( \pi \) is not of the form given in the statement of the proposition. By the classification in §2.7, it follows that \( \pi \) is of type (2.7SC–2), (2.7SC–3), or (2.7SC–4).

Let \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \in C \subset G_y \) have regular image \( \tilde{\gamma} \) in \( G_y \). Let \( \gamma' \in G_z \) be the conjugate of \( \gamma \) by \( \nu \) (see §2.3). Then \( \gamma \in G^{\text{reg}} \), and \( \gamma \) lies in a unique maximal parahoric by Lemma 2.1, namely \( G_y \). Also, the image \( \tilde{\gamma}' \) of \( \gamma' \) in \( G_z \) is regular elliptic, so that \( \gamma' \) is not contained in any parahoric other than \( G_z \). We note that \( \gamma \) and \( \gamma' \) are stably conjugate elements of \( G \) that are not conjugate in \( G \).

If \( \pi \) is of type (2.7SC–2) or (2.7SC–3), then \( \pi \) is compactly induced from the inflation \( \sigma \to G_y \) of a non-cubic cuspidal representation \( \sigma \) of \( G_y \). Thus \( \theta_\pi(\gamma') = 0 \) by Proposition 7.1 since \( \gamma' \) is not contained in any conjugate of \( G_y \). On the other hand, since the only conjugate of \( G_y \) containing \( \gamma \) is \( G_y \), \( \theta_\pi(\gamma') = \theta_\pi(\tilde{\gamma}) \) by Proposition 7.1.

Suppose that \( \pi \) is of type (2.7SC–3), i.e., \( \sigma = \tau \cdot (\eta \circ \det) \) where \( \tau \) is the cuspidal unipotent representation of \( G_y \) and \( \eta \in \text{Hom}(k_E^1, \mathbb{C}^\times) \). Then

\[
\theta_\pi(\tilde{\gamma}) = 2(\eta \otimes \eta \otimes \eta)(\tilde{\gamma}) \neq 0
\]

by [15, p. 31]. On the other hand, suppose that \( \pi \) is of type (2.7SC–2), i.e., the character of \( \sigma \) is \( -R_C^G \varphi \), where \( \varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \) is a character of \( C \) in general position. Then, by [27, 6.9],

\[
\theta_\pi(\tilde{\gamma}) = -\sum_{w \in W_{k^F}/(G_y)} w(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(\tilde{\gamma}).
\]

An easy application of the character theory of abelian groups shows that there is some \( \gamma \) of above type for which this sum does not vanish. Thus if \( \pi \) is of type (2.7SC–2) or (2.7SC–3), then \( \pi \) is not stable. Similarly, if \( \pi \) is compactly induced from \( G_z \) (i.e., \( \pi \) is of type (2.7SC–4)), then one can find stably conjugate \( \gamma \) and \( \gamma' \) such that \( \theta_\pi(\gamma) = 0 \), but \( \theta_\pi(\gamma') \neq 0 \). Hence \( \pi \) is again not stable. \( \square \)

4.3. **Non-singleton \( L \)-packets containing supercuspidals.** Let \( \gamma \in C \) be a regular element of \( G \) whose image \( \tilde{\gamma} \) in \( C \) is a regular element of \( G_y \). Let \( \gamma' \in C' \) be the conjugate of \( \gamma \) by \( \nu \). Since \( \gamma \in G \) and \( \tilde{\gamma} \in G \) are regular elliptic, Lemma 2.1 implies that \( \gamma \) lies in a unique maximal parahoric subgroup, namely \( G_y \). Similarly, \( \gamma' \) is not contained in any parahoric subgroup other than \( G_z \). The following lemma then follows easily from Proposition 7.1, [27, 6.9], and [15, p. 31].
Lemma 4.3. Let $\pi$ be a supercuspidal representation of $G$ of depth zero. Then, in the notation of §2.7,

$$
\theta_\pi(\gamma) = \begin{cases} 
-\sum_{w \in W_{\mu}(\mathcal{L}_G)} \omega(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(\overline{\gamma}) & \text{if } \pi \text{ is of type (2.7SC–2)}, \\
2(\eta \otimes \eta \otimes \eta)(\overline{\gamma}) & \text{if } \pi \text{ is of type (2.7SC–3)}, \\
0 & \text{if } \pi \text{ is of type (2.7SC–4)},
\end{cases}
$$

$$
\theta_{\pi'}(\gamma') = \begin{cases} 
0 & \text{if } \pi \text{ is of type (2.7SC–2)}, \\
0 & \text{if } \pi \text{ is of type (2.7SC–3)}, \\
-\sum_{w \in W_{\mu}(\mathcal{L}_G)} \omega(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(\overline{\gamma}) & \text{if } \pi \text{ is of type (2.7SC–4)}.
\end{cases}
$$

There are two types of non-singleton $L$-packets containing a supercuspidal representation of $G$ of depth zero as discussed in §2.7; namely, the non-supercuspidal $L$-packets of size two and the supercuspidal $L$-packets of size four. An $L$-packet II of the former type consists of the unique square-integrable constituent $\pi = \pi^2(\lambda)$ (see §2.7) of a reducible principal series of type (2.7PS–2) together with a corresponding supercuspidal representation $\pi^s = \pi^s(\lambda)$. Here $\lambda$ is a depth-zero character of $M$ such that $\lambda_{\mid F^\times} = \omega_{E/F} \cdot \psi^{-1}$. Recall the characters $\lambda_1 \in \text{Hom}(E^\times, \mathbb{C}^\times)$ and $\lambda_2 \in \text{Hom}(E^1, \mathbb{C}^\times)$ determined by $\lambda$ according to (2.7.1). Let $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ denote the associated characters of $k^+_E$ and $k^+_E$, respectively.

Proposition 4.4. Let $\lambda$ be a depth-zero character of $M$ such that $\lambda_{\mid F^\times} = \omega_{E/F} \cdot \psi^{-1}$. Let $\tilde{\lambda}_1'$ denote the character of $k^+_E$ such that $\tilde{\lambda}_1' \circ \mathcal{N} = \tilde{\lambda}_1$.

(i) If $\lambda_1$ is trivial on $\mathcal{O}_E^\times$, then $\pi^s(\lambda) = \text{ind}^G_{G_y} \sigma$, where $\sigma$ is the inflation to $G_y$ of the representation $\tau \cdot (\tilde{\lambda}_2 \circ \text{det})$ of $G_y$.

(ii) If $\lambda_1$ is nontrivial on $\mathcal{O}_E^\times$, then $\pi^s(\lambda) = \text{ind}^G_{G_z} \sigma$, where $\sigma$ is the inflation to $G_z$ of the representation of $G_z$ whose character is $-R^G_{\mathcal{Z}}(\tilde{\lambda}_2 \otimes \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_1 \tilde{\lambda}_2)$.

Proof. We determine the supercuspidal representation $\pi^s = \pi^s(\lambda)$ by computing its character at certain regular elliptic elements of $G$. Recall that the irreducible constituents of $\text{ind}^G_{G_y}(\lambda)$ are $\pi^2(\lambda)$ and a non-tempered representation $\pi^0(\lambda)$ and that the set $\Pi' = \{\pi^s(\lambda), \pi^0(\lambda)\}$ is an $A$-packet of $G$. Then $\Pi'$ is the endoscopic lift from $H$ to $G$ of the character

$$
\xi = (\mu \lambda_2 \circ \text{det}_{U(1,1)}) \otimes \lambda_2
$$

of $H$, where $\mu \circ \mathcal{N} = \lambda_1 \cdot \mid E^1/\omega_{E'/E} \mid$ [23, §12.2, §13.1], and $E'$ is an unramified quadratic extension of $E$. Let $\omega$ be the central character of the elements of $H$ and let $f \in C(G, \omega)$. By [23, Thm. 13.1.1, Prop. 13.1.2],

$$
\theta_{\pi^s}(f) + \theta_{\pi^0}(f) = \theta_\xi(f^H),
$$

where $f \mapsto f^H$ is the endoscopic transfer from $G$ to $H$ (see [23, §4.3]). Thus

$$
\theta_{\pi^s} = \theta_\xi - \theta_{\pi^0},
$$

where $\theta_\xi$ is the distribution on $G$ that arises from $\theta_\xi$ via endoscopy. The same equation holds for the functions on $G^{\text{reg}}$ that represent these distributions. Let $\gamma$ be an element of $C$ whose image $\overline{\gamma}$ in $C$ is regular. Let $\gamma' \in C'$ be the conjugate of
γ by ν and let \( \tilde{\gamma}' \) be its image in \( C' \). In order to determine \( \pi^s \), we will evaluate the right-hand side of (4.3.2) at \( \gamma \) if \( \lambda_1|_{\Omega_E^\kappa} \) is trivial and at \( \gamma' \) if \( \lambda_1|_{\Omega_E^\kappa} \) is nontrivial.

First we compute \( \theta^G_\xi(\gamma) \) and \( \theta^G_\xi(\gamma') \). By [23, Lemma 12.5.1] and the particular form of \( \gamma \),

\[
\theta^G_\xi(\gamma) = \sum_{w \in W_F(C', H) \setminus W_F(C, G)} \kappa(c_w)\xi(w\gamma),
\]

where \( c_w \) is the class in

\[
D(C/F) := \ker\{H^1(F, C) \to H^1(F, G)\}
\]

represented by the cocycle \( \{s(w)w^{-1}\} \) (\( s \in \text{Gal}(\overline{F}/F) \)) and \( \kappa \) is the element of the dual of \( D(C/F) \) corresponding to the endoscopic group \( H \). Since \( W_F(C, G) = W(C, G) \), \( \kappa(c_w) = 1 \) for all \( w \in W_F(C', H) \setminus W_F(C, G) \). Since \( W_F(C, G) \cong S_3 \) while \( |W_F(C, H)| = 2 \), we obtain

\[
\theta^G_\xi(\gamma) = \xi((\gamma_1, \gamma_2, \gamma_3)) + \xi((\gamma_3, \gamma_1, \gamma_2)) + \xi((\gamma_2, \gamma_3, \gamma_1)).
\]

Evaluating this when \( \lambda_1|_{\Omega_E^\kappa} \) is trivial and using (4.3.1), we get

\[
3(\lambda_2 \otimes \lambda_2 \otimes \lambda_2)(\gamma) = 3(\lambda_2 \otimes \lambda_2 \otimes \lambda_2)(\tilde{\gamma}).
\]

As in the preceding paragraph,

\[
\theta^G_\xi(\gamma') = \sum_{w \in W_F(C', H) \setminus W_F(C', G)} \kappa(c_w)\xi(w\gamma'),
\]

where \( c_w \) is now the class in \( D(C'/F) \) represented by \( \{s(w)w^{-1}\} \). In this case \( W_F(C', G) \cong S_3 \) and \( |W_F(C', H)| = 2 \). Then \( \kappa(c_1) = 1 \), and an easy calculation shows that if \( w \) represents a nontrivial coset in \( W_F(C', H) \setminus W_F(C', G) \), then \( \kappa(c_w) = -1 \). Thus

\[
\theta^G_\xi(\gamma') = \xi((\gamma_1, \gamma_2, \gamma_3)) - \xi((\gamma_3, \gamma_1, \gamma_2)) - \xi((\gamma_2, \gamma_3, \gamma_1)).
\]

We evaluate this when \( \lambda_1|_{\Omega_E^\kappa} \) is nontrivial. Using (4.3.1), we obtain

\[
(\lambda_1\lambda_2 \otimes \lambda_2 \otimes \lambda_1\lambda_2)(\gamma) = (\lambda_2 \otimes \lambda_1\lambda_2 \otimes \lambda_1\lambda_2)(\gamma) - (\lambda_1\lambda_2 \otimes \lambda_1\lambda_2 \otimes \lambda_2\lambda_2)(\gamma).
\]

It remains to evaluate \( \theta_{\pi^n} \) at \( \gamma \) and \( \gamma' \). Since \( \gamma \) and \( \tilde{\gamma} \) are regular elliptic and \( \gamma \in G_y \), \( y \) is the unique fixed point of \( \gamma \) in \( B \) by Lemma 2.1. Then [24, Lemma III.4.10, Theorem III.4.16] implies that

\[
(\pi^n)^{G_{y^+}}(\gamma) = \text{trace}(\gamma)((\pi^n)^{G_{y^+}}).
\]

The analogous formula holds for \( \gamma' \) and \( z \). Hence we must determine \( (\pi^n)^{G_{y^+}} \) and \( (\pi^n)^{G_{y^+}} \).

Recall that \( \pi^n(\lambda) \) and \( \pi^2(\lambda) \) are the irreducible constituents of \( \text{ind}^{G_{y^+}}_B\lambda \). Let \( \hat{\lambda} \) be the character of \( M \) determined by \( \lambda \). Since \( G = G_y B \), we have that for any \( x \in \mathcal{F} \),

\[
\text{Res}_{G_y}\text{ind}^{G_{y^+}}_B\lambda = \text{ind}^{G_{y^+}}_{B \cap G_y}\lambda = \text{ind}^{G_{y^+}}_B\text{ind}^{G_{y^+}}_B\lambda,
\]

which contains \( \rho_y := \text{ind}^{G_{y^+}}_B\lambda \), the inflation to \( G_y \) of the representation \( \tilde{\rho}_y := \text{ind}^{G_{y^+}}_B\lambda \). Since \( \rho_y \) is trivial on \( G_{y^+} \), this implies that the space of \( G_{y^+} \)-fixed vectors in \( \text{ind}^{G_{y^+}}_B\lambda \)
contains \( \rho_y \). Moreover, by Mackey’s theorem and Frobenius reciprocity,

\[
\text{Hom}_{G_{\psi^{+}}} (1, \text{Res}_{G_{\psi^{+}}} \text{ind}^G_{B} \lambda) = \text{Hom}_{G_{\psi^{+}}} \left( 1, \bigoplus_{g \in G_{\psi^{+}} \setminus G/B} \text{ind}^G_{B} \psi^{+} \left( g \lambda \right) \right) \\
= \bigoplus_{g \in G_{\psi^{+}} \setminus G/B} \text{Hom}_{B \cap G_{\psi^{+}}} (1, \lambda g) \\
= \bigoplus_{g \in G_{\psi^{+}} \setminus G/B} \text{Hom}_{B \cap G_{\psi^{+}}} (1, 1).
\]

The dimension of this space is \(|G_{\psi^{+}} \setminus G/B|\), which (since \( G = G_{\psi}B \)) is equal to

\(|G_{\psi^{+}} \setminus G/B \cap G_{\psi^{+}}| = |G_{\psi}/B_{\psi}| = \dim \rho_y,\)

Hence the space of \( G_{\psi^{+}} \)-fixed vectors in \( \text{ind}^G_{B} \lambda \) is isomorphic to \( \rho_y \).

Since the vertex \( z \) is special, the Iwasawa decomposition \( G = G_zB \) holds, where \( B \) is the Borel subgroup opposite \( B \) with respect to \( M \). Then an argument similar to that in the preceding paragraph shows that, as a representation of \( G_z \), the space of \( G_z \)-fixed vectors in \( \text{ind}^G_{B} \lambda \) is isomorphic to \( \tilde{\rho}_z \equiv \text{ind}^G_{B} \lambda \).

Now let \( v \) equal \( y \) if \( \lambda_1 \mid \Omega_{\zeta}^x \) is trivial or \( z \) if \( \lambda_1 \mid \Omega_{\zeta}^x \) is nontrivial. Let \( \pi \) be either \( \pi^2 \) or \( \pi^n \). By [21, Thm. 5.2], for \( x \in F \), \( (G_z, \lambda \mid M_0) \) is a \( K \)-type contained in \( \pi \) (where we have identified \( G_z \mid G_{\psi^{+}} \) and \( M_0/M_0^{+} \)). Thus, as a representation of \( B_v \), \( \pi_{G_{\psi^{+}}} \) contains the character \( \lambda \) of \( B_v \). By Frobenius reciprocity, \( \pi_{G_{\psi^{+}}} \) contains \( \text{ind}^G_{B} \lambda \) isomorphic to \( \tilde{\rho}_z \). Since \( \lambda \) extends to a character of \( G_v \), \( \tilde{\rho}_v \) is reducible with two irreducible constituents. Replacing \( \lambda \) by a Weyl conjugate if necessary, we may assume that \( \pi^2 \) is a subrepresentation of \( \text{ind}^G_{B} \lambda \), so that we have the exact sequence

\[
0 \longrightarrow \pi^2 \longrightarrow \text{ind}^G_{B} \lambda \longrightarrow \pi^n \longrightarrow 0.
\]

Taking \( G_{\psi^{+}} \)-fixed vectors, we obtain the exact sequence

\[
0 \longrightarrow (\pi^2)^{G_{\psi^{+}}} \longrightarrow \tilde{\rho}_v \longrightarrow (\pi^n)^{G_{\psi^{+}}} \longrightarrow 0
\]

of representations of \( G_v \). It follows that as a representation of \( G_v \), \( \pi_{G_{\psi^{+}}} \) is an irreducible constituent of \( \tilde{\rho}_v \).

According to \( \S 2.5 \), the irreducible constituents of \( \tilde{\rho}_v \) are \( n \)-dimensional representations \( \psi \) and the representation \( \text{St}_{G_v}(\psi) \). Here

\[
\psi = (\lambda_1^{\prime} \circ \det_{U(1,1)} \circ p_v) \cdot (\lambda_2 \circ \det_{G_v}),
\]

where \( p_v : G_v \longrightarrow U(1,1) \) is trivial if \( v = y \) or the projection onto the \( U(1,1) \) factor of \( G_z \equiv U(1,1) \times U(1) \) if \( v = \tilde{z} \). Suppose that \( (\pi^2)^{G_{\psi^{+}}} \equiv \psi \). Then \( G_v \) acts via the character \( \psi \) on any nonzero vector \( u \in (\pi^2)^{G_{\psi^{+}}} \). Let \( (\pi^2)^{\psi} \) be the contragredient representation of \( \pi^2 \). Then \( G_v \) acts via \( \psi^{-1} \) on any nonzero vector \( u' \in ((\pi^2)^{\psi})^{G_{\psi^{+}}} \). An easy computation shows that the matrix coefficient \( c_{u,u'} \) is not square-integrable. It follows that \( u \notin \pi^2 \) and hence that \( (\pi^n)^{G_{\psi^{+}}} \equiv \psi \). Thus, if \( \lambda_1 \mid \Omega_{\zeta}^x \) is trivial, then from (4.3.6),

\[
\theta_{\pi^n}(\gamma) = \text{trace}(\gamma(\pi^n)^{G_{\psi^{+}}}) = \psi(\tilde{\gamma}) = (\lambda_2 \otimes \lambda_2 \otimes \lambda_2)(\tilde{\gamma}).
\]

On the other hand, if \( \lambda_1 \mid \Omega_{\zeta}^x \) is nontrivial, then from (4.3.6),

\[
\theta_{u^n}(\gamma') = \text{trace}(\gamma'(\pi^n)^{G_{\psi^{+}}}) = \psi(\tilde{\gamma}') = (\lambda_1^{\prime} \lambda_2 \otimes \lambda_2 \otimes \lambda_1^{\prime} \lambda_2)(\tilde{\gamma}).
\]
Combining these calculations with (4.3.4), (4.3.5) and (4.3.2), we find that if \( \lambda_1|_{\mathcal{O}_E^x} \) is trivial,
\[
\theta_{x^*}(\gamma) = 2(\bar{\lambda}_2 \otimes \bar{\lambda}_2)(\gamma),
\]
while if \( \lambda_1|_{\mathcal{O}_E^x} \) is nontrivial
\[
\theta_{x^*}(\gamma) = -((\lambda_2 \otimes \lambda_1 \lambda_2 \otimes \lambda_1 \lambda_2)(\gamma) - (\bar{\lambda}_1 \bar{\lambda}_2 \otimes \bar{\lambda}_1 \bar{\lambda}_2 \otimes \bar{\lambda}_2)(\gamma)).
\]

Suppose that \( \lambda_1|_{\mathcal{O}_E^x} \) is trivial. Since \( x^* \) is a depth-zero supercuspidal representation, Lemma 4.3 implies that \( \theta_{x^*}(\gamma) \) is equal to the evaluation at \( \bar{\gamma} \) of a linear combination \( \mu \) of characters of \( C \) depending only on \( x^* \). Letting \( \gamma \) vary over all elements of \( C \) that are regular in \( G \) and that have regular image \( \tau \) in \( G_y \), we obtain from (4.3.7) that \( \mu = 2(\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2) \) on the set of regular elements of \( C \). By Cor. 2.6, it must be the case that \( \mu = 2(\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2) \). By the linear independence of characters of \( C \), \( \mu \) must have the character \( \lambda_2 \otimes \lambda_2 \otimes \lambda_2 \) as a summand. Hence, by Lemma 4.3, \( x^* \) must be equivalent to \( \text{ind}\_\gamma G_y^\lambda \sigma \), where \( \sigma \) is the inflation to \( G_y \) of \( \tau \cdot (\bar{\lambda}_2 \circ \det) \). This proves (i). A similar argument with \( \gamma' \) replacing \( \gamma \) proves (ii).

We now determine the \( L \)-packets of \( G \) of size 4. Fix distinct characters \( \chi_1, \chi_2, \) and \( \chi_3 \) of \( k^1_E \). Let \( \chi \) be the character \( \chi_1 \otimes \chi_2 \otimes \chi_3 \) of \( k^1_E \times k^1_E \times k^1_E \). Define regular characters \( \chi^{(1)}, \chi^{(2)}, \) and \( \chi^{(3)} \) of \( C' \) by
\[
\chi^{(1)} = \chi \\
\chi^{(2)} = \chi_2 \otimes \chi_3 \otimes \chi_1 \\
\chi^{(3)} = \chi_3 \otimes \chi_1 \otimes \chi_2.
\]

Note that each \( \chi^{(i)} \) is equal to \( w \chi \) for some \( w \in W_{k_E}(\mathcal{C}', \mathcal{G}_e) \). Let \( \sigma \) be the inflation to \( G_y \) of the cuspidal representation \( \sigma \) of \( G_y \) with character \( \bar{R}^G_C \chi \). For \( i = 1, 2, 3 \), let \( \sigma_i \) be the inflation to \( G_z \) of the cuspidal representation \( \sigma_i \) of \( G_z \) with character \( \bar{R}^G_z \chi^{(i)} \). Then \( \sigma_1, \sigma_2, \sigma_3 \) are distinct by [27, p. 139]. Define \( \pi_0 = \text{ind}\_\gamma G_y^\lambda \sigma \) and \( \pi_i = \text{ind}\_\gamma G_y^\lambda \sigma_i \) (\( i = 1, 2, 3 \)). By [21], these representations are inequivalent supercuspidals of depth zero. For \( v = y \) or \( z \), let \( \sigma_v \) be the inflation to \( H_v \) of the cuspidal representation of \( H_v \) with character \( \bar{R}^H_T \chi \), where \( T = C \) if \( v = y \), and \( T = C' \) if \( v = z \). Define \( \rho_v = \text{ind}\_\gamma H_v \sigma_v \). Then \( \rho_y \) and \( \rho_z \) are inequivalent but conjugate by an element of \( \text{PGL}_2(F) \times \{1\} \), and hence \( \{\rho_y, \rho_z\} \) is an \( L \)-packet for \( H \).

**Proposition 4.5.** The set \( \{\pi_0, \pi_1, \pi_2, \pi_3\} \) is an \( L \)-packet for \( G \) and is the endoscopic transfer of \( \{\rho, \rho'\} \).

**Proof.** Let \( R = \{\rho_y, \rho_z\} \) and let \( \Pi \) be the transfer of \( R \) from \( H \) to \( G \). Then \( \Pi \) has size four by [23, Prop. 13.1.2]. Let \( \pi'_0, \pi'_1, \pi'_2, \pi'_3 \) be the elements of \( \Pi \). Then the \( \pi'_i \) are supercuspidal by [23, Prop. 13.1.3(b)]. That they have depth zero follows from our assumption (see the Introduction) that the transfer preserves depth. Set \( \theta_R = \theta_{\pi'_0} + \theta_{\pi'_3} \). Let \( \theta_R^G \) be the endoscopic transfer of \( \theta_R \) from \( H \) to \( G \). It follows from [23, Thm. 13.1.1, Prop. 13.1.3, Lemma 12.7.2] that
\[
\theta_R^G = \theta_{\pi'_0} + \theta_{\pi'_3} - \theta_{\pi'_2} - \theta_{\pi'_2}
\]
for some ordering of the \( \pi'_i \). Letting \( \gamma \) and \( \gamma' \) be as in Proposition 4.4. We will compute \( \theta_R(\gamma) \) and \( \theta_R(\gamma') \) to determine the \( \pi'_i \).
Let \( \gamma^* \) be either \( \gamma \) or \( \gamma' \), and correspondingly let \( T \) be either \( C \) or \( C' \). According to [23, Lemma 12.5.1], using the notation in the proof of Proposition 4.4,

\[
\theta^G_R(\gamma^*) = \sum_{w \in W_F(T,H) \setminus W_F(T,G)} \kappa(c_w) \theta^R(w \gamma^*).
\]

As in the proof of Proposition 4.4, if \( \gamma^* = \gamma \), then \( \kappa(c_w) = 1 \) for all \( w \in W_F(C,H) \setminus W_F(C,G) \), while if \( \gamma^* = \gamma' \), then \( \kappa(c_1) = 1 \) and \( \kappa(c_w) = -1 \) if \( w \) represents a nontrivial coset in \( W_F(C',H) \setminus W_F(C',G) \). Since \( \gamma^* \in H \) and \( \gamma' \in H \) are regular elliptic, \( \gamma^* \) lies in a unique maximal parahoric subgroup \( H_v \) of \( H \) by Lemma 2.1 (where \( v = y \) if \( \gamma^* = \gamma \), and \( v = z \) if \( \gamma^* = \gamma' \)). Let \( u \) be either \( y \) or \( z \). It follows from Proposition 7.1 and [27, 6.9] that

\[
\theta_{\mu_u}(w \gamma^*) = \begin{cases} 
- \sum_{w \in W_{k_F}(T,H)} u w \chi(\tilde{\gamma}) & \text{if } u = v \\
0 & \text{if } u \neq v,
\end{cases}
\]

where we identify \( W_{k_F}(T,H) \) with \( W_F(T,H) \subset W_F(T,G) \). Hence

\[
\theta^G_R(\gamma) = - \sum_{w \in W_{k_F}(C,G)} w \chi(\tilde{\gamma}),
\]

\[
\theta^G_R(\gamma') = - \sum_{w \in W_{k_F}(C,G)} d_w w \chi(\tilde{\gamma}),
\]

(4.3.9)

where \( d_w = 1 \) if \( w \in W_{k_F}(C,H) \) and \( d_w = -1 \) otherwise.

As observed in the proof of Proposition 4.5, Lemma 4.3 implies that \( \theta_{\mu_1}(\gamma) \) is equal to the evaluation at \( \tilde{\gamma} \) of a linear combination \( \mu_1 \) of characters of \( C \) depending only on \( \pi'_i \). Therefore, evaluating (4.3.8) at all \( \gamma \) of the above type and using (4.3.9), we obtain

\[
- \sum_{w \in W_{k_F}(C,G)} w \chi = \mu_0 + \mu_1 - \mu_2 - \mu_3
\]

on the set of regular elements of \( C \). Then, by Cor. 2.6, this equation must hold at all elements of \( C \). It follows from Lemma 4.3 and the linear independence of characters of \( C \) that, after possibly reordering, \( \pi'_0 \) must be equivalent to \( \pi_0 \) and that the other elements of the \( L \)-packet must be induced from \( G_2 \). Evaluating (4.3.8) at \( \gamma' \) of the above type and using a similar argument, we obtain that \( \pi'_1 \equiv \pi_1 \) and, up to reordering, \( \pi'_i \equiv \pi_i \) for \( i = 2, 3 \).

\( \square \)

5. Explicit base change for \( G \)

5.1. Packets consisting of principal series constituents.

**Proposition 5.1.** Let \( \lambda \in \text{Hom}(M, \mathbb{C}^\times) \).

(i) If \( \text{ind}^G_B \lambda \) is irreducible and \( \text{ind}^G_B \lambda' \) is irreducible, then the base change lift of the \( L \)-packet \( \{ \text{ind}^G_B \lambda \} \) is \( \text{ind}^G_B \lambda' \).

(ii) If \( \text{ind}^G_B \lambda \) is irreducible but \( \text{ind}^G_B \lambda' \) is reducible, then \( \lambda_1|_{F^\times} = | \frac{E}{F} \lambda_2 \), and the base change lift of \( \{ \text{ind}^G_B \lambda \} \) is \( \text{ind}^G_P \left( \langle \lambda_1 \lambda_2 \rangle : | \frac{E}{F} \lambda_2 \rangle \circ \det_{GL(2)} \right) \).

(iii) If \( \lambda_1 = | \frac{E}{F} \lambda_1 \), then the lift of the \( L \)-packet comprising the one-dimensional constituent \( \psi = \lambda_2 \circ \det_{G_E} \) (respectively, the Steinberg constituent \( \text{St}_G(\psi) \)) of
Proof. Cases (i), (iii), and (iv) follow from [23] (Prop. 4.10.2 and the paragraph before Theorem 13.2.1). To prove case (ii), note that up to the action of the Weyl group, we may assume that $\lambda$ is positive with respect to $B$. The paragraph before Theorem 13.2.1 in [23] then implies that the base change lift of $\{\text{ind}_{B}^{G}\lambda\}$ is the Langlands quotient of $\text{ind}_{B}^{G}\lambda$. This quotient is the desired representation. □

5.2. Stable supercuspidal representations. Suppose $\pi$ is a depth-zero, stable, supercuspidal representation of $G$. From Prop. 4.2, $\pi^{G_{\psi}}$ contains the inflation $\sigma$ of a cubic cuspidal representation $\tilde{\sigma}$ of $G \cong G_{y}$. Then Figure 1 illustrates how to construct representations $\tilde{\pi}$ and $\tilde{\pi}'$ of $\tilde{G} \Gamma$. We can describe base change for $\pi$ explicitly by showing that $\tilde{\pi}$ and $\tilde{\pi}'$ are equivalent, provided that the extensions from $\tilde{G}$ to $\tilde{G} \Gamma$ and from $\tilde{G}$ to $\tilde{G} \Gamma$ are chosen in compatible ways.

Remark 5.2. Recall the Cartan decomposition for $\tilde{G}$: The diagonal subgroup $\tilde{M}$ determines a root system $\Phi$ for $\tilde{G}$, and the Borel subgroup $\tilde{B}$ determines a positive root system $\Phi^{+}$ inside $\Phi$. Let $\tilde{M}^{+}$ denote the set of all $m \in \tilde{M}$ such that $\alpha(m)$ has positive valuation for all $\alpha \in \Phi^{+}$. Then

$$\tilde{G} = \bigcup_{m \in \tilde{M}^{+}} \tilde{G}_{y} m \tilde{G}_{y}.$$ 

Moreover, $m, m' \in \tilde{M}^{+}$ represent the same double coset if and only if $m' \in m \tilde{M}_{0}$.

Lemma 5.3. Every conjugate of $\tilde{Z} \tilde{G}_{y} \Gamma$ in $\tilde{G} \Gamma$ is of the form $g^{m}(\tilde{Z} \tilde{G}_{y} \Gamma)$, where $g \in \tilde{G}_{y}, m \in \tilde{M}^{+}$.

Proof. The normalizer of $\tilde{Z} \tilde{G}_{y} \Gamma$ in $\tilde{G} \Gamma$ is $\tilde{Z} \tilde{G}_{y} \Gamma$ itself. Therefore, the conjugates of $\tilde{Z} \tilde{G}_{y} \Gamma$ correspond to the cosets in $\tilde{G} \Gamma / \tilde{Z} \tilde{G}_{y} \Gamma \cong \tilde{G} / \tilde{Z} \tilde{G}_{y}$. The lemma now follows from the Cartan decomposition. □

Remark 5.4. Recall that an inner automorphism of $\tilde{G}$ acts on the extended Dynkin diagram either trivially or via a rotation. Thus, if $g \in \tilde{G}$ stabilizes the alcove $\Delta \subset \tilde{B}^{\text{red}}$ and fixes some point in the closure of $\Delta$ not equal to the barycenter of $\Delta$, then $g$ must fix $\Delta$ pointwise.

In the next result, we use the fact that $\tilde{G}_{y}$ (in addition to being a quotient of $\tilde{G}_{y}$) is a quotient of $\tilde{Z} \tilde{G}_{y}$.

Proposition 5.5. Let $\sigma$ be the inflation to $G_{y}$ of a cubic cuspidal representation $\tilde{\sigma}$ of $G_{y}$, and let $\tilde{\sigma}$ be the Shintani lift of $\tilde{\sigma}$ from $G_{y}$ to $\tilde{G}_{y}$. Let $\pi = \text{ind}_{G_{y}}^{G_{\psi}} \sigma$. Then the base change lift of the $L$-packet $\{\pi\}$ is

$$\text{ind}_{\tilde{Z} \tilde{G}_{y}}^{\tilde{G}_{y}} \tilde{\sigma},$$

where $\tilde{\sigma}$ is the inflation to $\tilde{Z} \tilde{G}_{y}$ of $\tilde{\sigma}$. 

Proof. Let \( \tilde{\pi} \) be the base change lift of \( \{ \pi \} \) and let \( \tilde{\pi}' = \text{ind}_{\tilde{Z}G_y}^\tilde{G} \tilde{\sigma} \). Since \( \theta_{\pi} \) is stable by Proposition 4.2, \( \theta_{\tilde{\pi}, \varepsilon} \) is a stable \( \varepsilon \)-class function on \( \tilde{G}^{\varepsilon-\text{reg}} \) according to [23, §12.5]. By [21, Prop. 6.8], \( \tilde{\pi}' \) is a supercuspidal representation of \( \tilde{G} \) of depth zero. Also, \( \varepsilon \tilde{\pi}' \simeq \text{ind}_{\tilde{Z}G_y}^\tilde{G} \varepsilon \tilde{\sigma} \simeq \tilde{\pi}' \) since \( \tilde{\sigma} \) is \( \varepsilon \)-invariant as it is in the image of the Shintani lift. Since \( \tilde{\pi}' \) is \( \varepsilon \)-invariant, \( \tilde{\pi}' \) is the base change lift of a singleton supercuspidal \( L \)-packet \( \{ \pi' \} \) by [23, Prop. 13.2.2]. But then, as in the case of \( \tilde{\pi} \), \( \theta_{\tilde{\pi}', \varepsilon} \) is a stable \( \varepsilon \)-class function. Moreover, it is easily seen (under the assumption that \( \pi \) has depth zero) that the central characters of \( \tilde{\pi} \) and \( \tilde{\pi}' \) are identical. Furthermore, according to [23, §13.2], we may choose \( \tilde{\pi}(\varepsilon) \) and \( \tilde{\pi}'(\varepsilon) \) so that \( \theta_{\tilde{\pi}, \varepsilon} = \theta_{\pi} \circ N \) and \( \theta_{\tilde{\pi}', \varepsilon} = \theta_{\pi'} \circ N \) (see §2.1).

If \( \alpha_1 \) and \( \alpha_2 \) are stable \( \varepsilon \)-class functions on \( \tilde{G}^{\varepsilon-\text{reg}} \) that transform under \( \tilde{Z} \) via the same character, then the \( \varepsilon \)-elliptic inner product of \( \alpha_1 \) and \( \alpha_2 \) (see [23, §12.5]) is defined by

\[
\langle \alpha_1, \alpha_2 \rangle_{\varepsilon} = \sum_{T \in \mathcal{C}} |W_F(T, \tilde{G})|^{-1} \int_{\tilde{Z}T' \setminus \tilde{T}} D_G(N(\delta))^{\varepsilon} \alpha_1(\delta)\alpha_2(\delta) \, d\delta,
\]

where \( \mathcal{C} \) is a set of representatives for the stable conjugacy classes of elliptic Cartan subgroups of \( G \), \( \tilde{T} \) is the centralizer of the Cartan subgroup \( T \) in \( \tilde{G} \), \( \tilde{T}' \) is the kernel of the norm map on \( \tilde{T} \), and \( D_G \) is the discriminant.

By [23, Prop. 12.6.2], in order to prove that \( \tilde{\pi} \simeq \tilde{\pi}' \), it suffices to show that \( \langle \theta_{\tilde{\pi}', \varepsilon}, \theta_{\tilde{\pi}, \varepsilon} \rangle_{\varepsilon} \neq 0 \).

We will verify the non-vanishing of this inner product by showing that the two twisted characters agree on \( \tilde{T} \) for each \( T \in \mathcal{C} \).
By Proposition 4.2, the stability of $\pi'$ implies that $\pi'$, like $\pi$, is induced from the inflation $\sigma'$ to $G_y$ of a cubic cuspidal representation $\tilde{\sigma}'$ of $G_y$. Since $\tilde{\sigma}$ and $\tilde{\sigma}'$ both arise via Deligne-Lusztig induction from a cubic Cartan subgroup of $G_y$, these representations agree on unipotent elements of $G_y$ [27, 6.9]. It follows that $\theta_\pi$ and $\theta_{\pi'}$ agree on $G_{0^+}$. Since $\pi$ and $\pi'$ have the same central character, $\theta_\pi$ and $\theta_{\pi'}$ agree on $ZG_{0^+}$.

Let $T \in \mathcal{C}$, $\delta \in \widetilde{T} \cap \widetilde{G}^{\text{reg}}$, and $\gamma = N(\delta)$. If $\gamma \in ZT_{0^+}$, then
\[
\theta_{\pi,\varepsilon}(\delta) = \theta_\pi(\gamma) = \theta_{\pi'}(\gamma) = \theta_{\pi',\varepsilon}(\delta).
\]
We may therefore assume that $\gamma \in T \setminus ZT_{0^+}$.

If no conjugate of $\gamma$ is contained in $G_y$, then both $\theta_{\pi,\varepsilon}(\delta) = \theta_\pi(\gamma)$ and $\theta_{\pi',\varepsilon}(\delta) = \theta_{\pi'}(\gamma)$ vanish by Proposition 7.1. We may therefore assume that $\gamma \in G_y$.

We may assume that that $T$ is not of type (2.4–0), since such tori are not elliptic.

Suppose that $T$ is of type (2.4–1) or (2.4–2). As in the proof of Proposition 4.2, since $\gamma \in T \setminus ZT_{0^+}$, the semisimple part of the image $\bar{\gamma}$ of $\gamma$ in $G_y$ is not contained in a cubic torus of $G_y$. Therefore, by [27, 6.9], $\theta_\pi$ and $\theta_{\pi'}$ vanish on $\gamma$. Thus, again we have
\[
\theta_{\pi,\varepsilon}(\delta) = \theta_\pi(\gamma) = 0 = \theta_{\pi'}(\gamma) = \theta_{\pi',\varepsilon}(\delta)
\]
by Proposition 7.1.

Now suppose that $T$ is of type (2.4–3). Then there exist cubic extensions $L$ of $E$ and $K$ of $F$ such that $L = EK$ and $T \cong \text{Ker}(N_{L/K})$. In particular, we may identify $T_{0^+}$ with $\text{Ker}(N_{L/K}) \cap (1 + p_L)$ and $Z$ with $E^1$. We therefore have $ZT_{0^+} \cong E^1 [\text{Ker}(N_{L/K}) \cap (1 + p_L)]$. If $L/E$ is totally ramified, then $E^1 [\text{Ker}(N_{L/K}) \cap (1 + p_L)] = \text{Ker}(N_{L/K})$ so $T \setminus ZT_{0^+}$ is empty, and there is nothing to prove in this case. We may hence assume that $L/E$ is unramified. Since $T$ is determined only up to stable conjugacy, we may also assume that $T$ fixes the point $y$.

Let $\bar{\gamma}$ be the image of $\gamma$ in the cubic torus $T \subset G_y$. Since $\gamma \notin ZT_{0^+}$, $\bar{\gamma}$ is not central in $G_y$. Thus $\bar{\gamma}$ is regular elliptic, so by Lemma 2.1, $\gamma$ is contained in a unique parahoric subgroup of $G$, namely $G_y$. Thus
\[
\theta_{\pi,\varepsilon}(\delta) = \theta_\pi(\gamma) = \theta_\pi(\bar{\gamma})
\]
by Proposition 7.1. It suffices to show that $\theta_{\pi',\varepsilon}(\delta) = \theta_{\bar{\pi}}(\bar{\gamma})$.

Extend $\bar{\pi}'$ to a representation (also denoted $\bar{\pi}'$) of $\widetilde{G}\Gamma$ in a manner compatible with the choice of $\bar{\pi}'(\varepsilon)$ made in the beginning of the proof. Then
\[
\theta_{\pi',\varepsilon}(\delta) = \theta_{\bar{\pi}}(\bar{\delta}\varepsilon).
\]
As a representation of $\widetilde{G}\Gamma$,
\[
\bar{\pi}' \cong \text{ind}_{\widetilde{G}_\mathbb{F}_L}^{\widetilde{G}\Gamma} \bar{\sigma},
\]
where $\bar{\sigma}$ is extended compatibly from $\widetilde{G}_y$ to $\widetilde{G}_y\Gamma$. This extension determines an extension of $\bar{\sigma}$ to $\widetilde{G}_y\Gamma$, and we let $\theta_{\bar{\sigma}}$ be the corresponding twisted character.

By Proposition 7.1, to compute $\theta_{\bar{\pi}}(\bar{\delta}\varepsilon)$, we must determine which conjugates of $\widetilde{ZG}_y\Gamma$ contain $\bar{\delta}\varepsilon$. Since
\[
\widetilde{T}N\widetilde{T}_0 \cong K^\times O_L^\times = L^\times \cong T,
\]
and since $\theta_{\pi',\varepsilon}(\delta)$ only depends on $\delta$ modulo $\widetilde{T}N$, we may assume that $\delta \in \widetilde{T}_0$. Since $T$ fixes $y$, $\delta \in \widetilde{T}_0 \subset \widetilde{G}_y$ so $\delta \varepsilon \in \widetilde{G}_y\Gamma$. 
Now suppose that \( \delta \epsilon \) is also contained in another conjugate of \( \tilde{\mathcal{G}}_y \Gamma \). By Lemma 5.3, any such conjugate is of the form \( s^m(\tilde{\mathcal{G}}_y \Gamma) \), where \( g \in \tilde{\mathcal{G}}_y \) and \( m \neq 1 \) is in \( \tilde{M}^+ \). If \( \delta \epsilon \in s^m(\tilde{\mathcal{G}}_y \Gamma) \), then \( (s^m)^{-1}(\delta \epsilon) \in \tilde{\mathcal{G}}_y \Gamma \) so
\[
m^{-1} \delta \epsilon(m) \in \tilde{\mathcal{G}}_y,
\]
where \( \delta' = g^{-1} \delta \epsilon(g) \in \tilde{\mathcal{G}}_y \). Thus \( \delta \epsilon(m) \in m \tilde{\mathcal{G}}_y \) so \( \epsilon(m) \) and \( mc \) represent the same double coset in \( G_y / \tilde{G} / \tilde{G}_y \) for some \( c \in \tilde{Z} \). Since \( \epsilon(m) \), \( mc \in \tilde{M}^+ \), we have \( \epsilon(m) \in mc \tilde{M}_0 \) by Remark 5.2. It follows that \( \delta' \) is in \( m(\tilde{\mathcal{G}}_y) = \tilde{\mathcal{G}}_{my} \tilde{Z} \) as well as \( \tilde{\mathcal{G}}_y \).

Let \( \tilde{y} \) be the image of \( y \) in \( \tilde{B}^{\text{red}} \). Then \( \delta' \) fixes \( \tilde{y} \) and \( m \tilde{y} \), hence fixes the line segment \( [\tilde{y}, m \tilde{y}] \) in \( \tilde{B}^{\text{red}} \). Since \( \epsilon(m) \in mc \tilde{M}_0 \), we have \( \epsilon(m \tilde{y}) = m \tilde{y} \) so that \( [\tilde{y}, m \tilde{y}] \) intersects the (open) triangle \( \Delta \subset \tilde{B}^{\text{red}} \) non-trivially. Hence \( \delta' \) stabilizes \( \Delta \). From Remark 5.4, \( \delta' \) must fix \( \Delta \) pointwise. Thus \( \delta' \) is contained in \( G_{\tilde{\mathcal{P}}} \tilde{Z} \), where \( G_{\tilde{\mathcal{P}}} \tilde{Z} \) is the standard upper-triangular Iwahori subgroup of \( \tilde{G} \). The image \( \tilde{\delta}' \) of \( \delta' \) in \( \tilde{G}_y \) is therefore contained in the Borel subgroup \( \tilde{B}_y \) of upper-triangular matrices in \( \tilde{G}_y \). Since \( \tilde{B}_y \) is \( \epsilon \)-invariant, \( \tilde{\delta}' \) is also contained in \( \tilde{B}_y \). Hence the eigenvalues of \( \delta' \epsilon(\delta') \) lie in \( k_{\tilde{L}}^1 \). But \( \delta' \epsilon(\delta') = \tilde{N}(\delta') = \tilde{N}(\tilde{g}^{-1} \delta \epsilon(\tilde{g})) = \tilde{g}^{-1} \tilde{g}, \) where \( \tilde{g} \) is the image of \( g \) in \( \tilde{G}_y \), so the eigenvalues of \( \delta' \epsilon(\delta') \) are the same as those of \( \tilde{\gamma} \). The eigenvalues of \( \tilde{\gamma} \), however, lie in \( k_{L}^1 \setminus k_{\tilde{L}}^1 \) since \( \tilde{\gamma} \) is a regular element of the cubic torus \( \tilde{T} \). This contradiction shows that \( \delta \epsilon \) is contained in a unique conjugate of \( \tilde{\mathcal{G}}_y \Gamma \), namely \( \tilde{\mathcal{G}}_y \Gamma \) itself.

We therefore have from Proposition 7.1 that
\[
\theta_{\tilde{\pi}', \epsilon}(\delta) = \theta_{\tilde{\pi}'}(\delta \epsilon) = \theta_{\tilde{\pi}', \epsilon}(\tilde{\delta}),
\]
But \( \tilde{\sigma} \) is the Shintani lift of \( \sigma \) (see [17]) so the last expression is equal to
\[
\pm \theta_{\tilde{\sigma}}(\tilde{\gamma}).
\]
(Here the twisted character \( \theta_{\tilde{\sigma}, \epsilon} \) as chosen above is not a priori equal to \( \theta_{\tilde{\sigma}} \circ \mathcal{N} \) since this choice is not necessarily the one that is compatible with the Shintani lifting. Nevertheless, it is at worst off by a sign by the discussion in 2.1.) At the same time
\[
\theta_{\tilde{\pi}', \epsilon}(\delta) = \theta_{\tilde{\pi}'}(\gamma) = \theta_{\tilde{\pi}'}(\tilde{\gamma}),
\]
so \( \theta_{\tilde{\pi}'}(\tilde{\gamma}) = \pm \theta_{\tilde{\sigma}}(\tilde{\gamma}) \). It is easily seen (e.g., from the character table in [15]) that there is no cubic cuspidal representation \( \tilde{\sigma}' \) of \( \tilde{G}_y \) satisfying \( \theta_{\tilde{\sigma}'}(\tilde{\gamma}) = -\theta_{\tilde{\sigma}}(\tilde{\gamma}) \) for all regular elements \( \tilde{\gamma} \) of cubic tori. Thus
\[
\theta_{\tilde{\pi}', \epsilon}(\delta) = \theta_{\tilde{\sigma}}(\tilde{\gamma}),
\]
and the theorem follows. \( \square \)

5.3 Non-singleton \( L \)-packets containing supercuspidals.

**Proposition 5.6.** Let \( \lambda \) be a character of \( M \) of depth zero such that \( \lambda_1|_{\mathcal{E}^{0}} = \omega_{\mathcal{E}/\mathbb{F}}|_{\mathcal{E}^{0}} \). If
\[
\lambda = \lambda_1 \lambda_2 \in \mathbb{F}^{\mathbb{Z}}.
\]
(i) The base change lift of the \( L \)-packet \( \{ \pi^2(\lambda), \pi^s(\lambda) \} \) is
\[
\text{ind}_{\tilde{P}}^{\tilde{G}} \left( \text{St}_{\tilde{B}} \left( (\lambda_1 \lambda_2 | \|_{\mathcal{E}}^{1/2} \circ \det_{GL(2)}) \otimes \lambda_2 \right) \right).
\]
(ii) The base change lift of the $A$-packet $\{\pi^n(\lambda), \pi^s(\lambda)\}$ is
\[
\text{ind}_{B}^{G} \left( (\lambda_1 \lambda_2 | \cdot |_{E}^{1/2} \circ \det_{GL(2)}) \otimes \tilde{\lambda}_2 \right).
\]
Moreover, the above two base change lifts are precisely the irreducible constituents of the principal series representation $\text{ind}_{B}^{G}(\lambda)$.

Note that the proposition has the same content if we restrict the choice of exponent in the hypothesis to be $+1$ (or to be $-1$).

Proof. This follows from [23, §§12–13]. More precisely, let $\xi$ be the character
\[
(\mu \lambda_2 \circ \det_{U(1,1)}) \otimes \lambda_2
\]
of $H$, where
\[
\mu \circ N = \lambda_1 | \cdot |_{E}^{1/2} \omega_{E'/E},
\]
$E'$ an unramified quadratic extension of $E$. (Here, we are identifying $H$ with $U(1,1)(F) \times U(1)(F)$.) Let $\rho = \text{St}_{H}(\xi)$. Then, by [23, Prop. 13.1.3(c)], the $L$-packet $\{\pi^2(\lambda), \pi^s(\lambda)\}$ on $G$ is the lift of the $L$-packet $\{\rho\}$ on $H$. It follows from [23, Prop. 13.2.2 (c)] that the base change lift of $\{\pi^2(\lambda), \pi^s(\lambda)\}$ is $\text{ind}_{B}^{G}(\rho')$, where $\rho'$ is the “primed” base change lift (see [23, §11.4]) of $\rho$ from $H$ to $\tilde{H}$. But by [23, §12.1],
\[
\rho' = \text{St}_{\tilde{H}}(\tilde{\xi}),
\]
where $\tilde{\xi}$ is the character
\[
(\omega_{E'/E}(\mu \lambda_2 \circ N) \circ \det_{GL(2)}) \otimes \tilde{\lambda}_2 = (\lambda_1 \lambda_2 | \cdot |_{E}^{1/2} \circ \det_{GL(2)}) \otimes \tilde{\lambda}_2.
\]
This proves (i), and (ii) follows analogously from [23, Prop. 13.1.3(d)].

The final statement follows from the proof of [23, Lemma 12.7.6].

Recall the notation of Proposition 4.5. Let $\Pi$ be the supercuspidal $L$-packet $\{\pi_0, \pi_1, \pi_2, \pi_3\}$, and let $R = \{\rho_p, \rho_\varphi\}$ be the $L$-packet of $H$ that transfers to $\Pi$.

**Proposition 5.7.** The base change lift of the $L$-packet $\Pi = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ is $\text{ind}_{B}^{G}(\chi^*)$, where $\chi^*$ is inflation to $\tilde{M} \cong E^x \times E^x \times E^x$ of
\[
\hat{\chi} = \hat{\chi}_1 \otimes \hat{\chi}_2 \otimes \hat{\chi}_3 \in \text{Hom}(\tilde{M}, C^x).
\]

Proof. This also follows from [23, §§12–13]. Note that since $E/F$ is unramified, $\tilde{M}$ is a quotient of $\tilde{M}$, so the definition of $\chi^*$ makes sense. Let $\tilde{\rho}'$ be the “primed” base change lift (see [23, §11.4]) of $R$ from $H$ to $\tilde{H}$. By [23, Prop. 13.2.2(c)], the base change lift of $\Pi$ is $\text{ind}_{B}^{G}(\tilde{\rho}')$. By [23, §12.1], $R$ is the transfer from $C$ to $H$ of some character $\varphi$ of $C$. Let $\theta^H_\varphi$ be the distribution on $H$ that arises from $\theta^H_\varphi = \varphi$ via endoscopy. Let $B'$ be a Borel subgroup of $H$ containing $\tilde{C}$. Then by [23, §12.1], $\tilde{\rho}'$ is the representation $\text{ind}_{B'}^{H}(\hat{\varphi})$. Hence the base change lift of $\Pi$ is
\[
\text{ind}_{B}^{G} \text{ind}_{B'}^{H}(\hat{\varphi}) = \text{ind}_{B'}^{H}(\hat{\varphi}).
\]

We now determine $\varphi$.

Since $R$ has depth zero, $\varphi$ must as well. Also,
\[
\theta^H_\varphi = \pm (\theta_\rho - \theta_\rho')
\]
by [23, Prop. 11.1.1(b)]. The same equation holds for the functions that represent these distributions. Let $\gamma$ be an element of $C$ whose image $\bar{\gamma} \in \mathbb{C}$ is regular in $H_y$. As computed in the proof of Proposition 4.5,

$$\theta_\mu(\gamma) = - \sum_{w \in W_{k_F}(C,H)} w \chi(\bar{\gamma}),$$

while $\theta_{\mu'}(\gamma) = 0$. Hence the evaluation of the right side of (5.3.1) at $\gamma$ is

$$\pm \sum_{w \in W_{k_F}(C,H)} w \chi(\bar{\gamma}).$$

The analogue of (4.3.3) for the transfer from $C$ to $H$ implies that

$$\theta^H_\varphi(\gamma) = \sum_{w \in W_F(C,H)} \varphi^H(w\gamma) = \sum_{w \in W_{k_F}(C,H)} w \bar{\varphi}(\bar{\gamma}),$$

where $\bar{\varphi}$ is the character of $C$ determined by $\varphi$. Using (5.3.1) and letting $\gamma$ vary over all elements of the above type, it follows that

$$\sum_{w \in W_{k_F}(C,H)} w \bar{\varphi} = \pm \sum_{w \in W_{k_F}(C,H)} w \chi,$$

on the set of regular elements of $C$, hence on all of $C$ by Cor. 2.6. By the linear independence of characters of $C$, it follows that $\bar{\varphi} = w \chi$ for some $w \in W_{k_F}(C,H)$. Since $\bar{\varphi}$ is in the image of the base change lifting from $C$ to $\bar{C}$, it follows from [23, §12.4] that $\bar{\varphi}$ is trivial on elements of $\bar{C}$ of the form $(\varpi^a, \varpi^b, \varpi^c)$. Since $\bar{\varphi}$ has depth zero, $\bar{\varphi}$ must be the inflation to $\bar{C}$ of $w \bar{\chi}$ for some $w$. Thus $w^{-1} \bar{\varphi}$ is the inflation to $\bar{C}$ of $\bar{\chi}$, where $w$ is viewed as an element of $W(\bar{C}, \bar{G})$. Moreover,

$$\bar{\pi} = \text{ind}_{\bar{B}}^{\bar{C}} \bar{\varphi} \cong \text{ind}_{\bar{B}}^{\bar{C}} w^{-1} \bar{\varphi}.$$

Finally, note that by conjugating by a suitable element, one can send $\bar{B}'$, $\bar{C}$, and $w^{-1} \bar{\varphi}$ respectively to $\bar{B}$, $\bar{M}$, and $\chi^*$. The theorem follows.

\[\Box\]

6. Compatibility of base change and $K$-types

In this section we prove the Main Theorem, as stated in §1. Throughout, $\Pi$ will denote an $L$-packet of $G$ and $\bar{\pi}$ the base change lift of $\Pi$.

6.1. Principal series $L$-packets. As in §5.1, suppose $\Pi$ consists entirely of constituents of the depth-zero principal series $\text{ind}_G^H \lambda$. Since each element of $\Pi$ has depth zero, $\text{ind}_G^H \lambda$ and hence $\lambda$ have depth zero by [21, Theorem 5.2]. It follows from [21] that for any $x \in \mathcal{F}$, $(G_x, \lambda|\mathcal{M}_0)$ is a $K$-type of each element of $\Pi$, where $\mathcal{M}$ is identified with $G_x/G_{x^+}$. Similarly, $(\bar{G}_x, \bar{\lambda})|\bar{\mathcal{M}}_0$ is a $K$-type of $\bar{\pi} = \text{ind}_G^\bar{G} \bar{\lambda}$ (see Proposition 5.1), where $\bar{G}_x/\bar{G}_{x^+}$ is identified with $\bar{M}$. Denote by $\tilde{\lambda}$ the character of $\mathcal{M}$ that inflates to $\lambda|\mathcal{M}_0$. Then $\tilde{\lambda}|\tilde{\mathcal{M}}_0$ is the inflation to $\tilde{\mathcal{M}}_0$ of the character $\bar{\lambda}$ of $\bar{\mathcal{M}}$. As required, this is the Shintani lift of $\lambda$ from $\mathcal{M}$ to $\bar{\mathcal{M}}$. 

6.2. Singleton supercuspidal L-packets. Now suppose that $\Pi$ is a singleton supercuspidal $L$-packet $\{\pi\}$ of depth zero. Then, by Proposition 4.2, $\pi$ is of the form $\text{ind}_{G_y}^G\sigma$, where $\sigma$ is the inflation to $G_y$ of a cubic cuspidal representation $\tilde{\sigma}$ of $G_y$. Then $(G_y, \sigma)$ is a $K$-type of $\pi$ by [21, Prop. 6.2]. Similarly, it follows from Proposition 5.5 and [21, Prop. 6.2] that $(\tilde{G}_y, \tilde{\sigma})$ is a $K$-type of $\tilde{\pi}$, where $\tilde{\sigma}$ is the inflation to $\tilde{G}_y$ of the Shintani lift $\tilde{\sigma}$ of $\tilde{\sigma}$ from $G_y$ to $\tilde{G}_y$. Hence the theorem holds in this case.

6.3. Supercuspidal L-packets of size four. Recalling the notation of Proposition 4.5, suppose that $\Pi = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ is a depth-zero supercuspidal $L$-packet. By [21, Prop. 6.2], $(G_y, \pi)$ and $(G_z, \pi_i)$ are $K$-types for $\pi_0$ and the $\pi_i$ $(i = 1, 2, 3)$, respectively.

According to Proposition 5.7, $\tilde{\pi}$ is the principal series representation $\text{ind}_{G_y}^G\chi^*$, where $\chi^*$ is the inflation to $\tilde{M} \cong E^* \times E^* \times E^*$ of

$$\tilde{\chi} = \tilde{\chi}_1 \otimes \tilde{\chi}_2 \otimes \tilde{\chi}_3 \in \text{Hom}(\tilde{M}, \mathbb{C}^*) .$$

View $\chi^*|_{\tilde{M}_0}$ as a character of $\tilde{G}_x$ (for any $x \in \mathcal{F}$) under the identification $\tilde{G}_x = \tilde{M}$. Then, by [21, Thm. 5.2], $(\tilde{G}_x, \chi^*|_{\tilde{M}_0})$ is a $K$-type for $\tilde{\pi}$. Since $\tilde{\pi}$ contains $(\tilde{G}_x, \chi^*|_{\tilde{M}_0})$, it follows that, as a representation of $\tilde{B}_y$, $\tilde{\pi}^{G_y}$ contains the character $\tilde{\chi}$ of $\tilde{B}_y$. Hence, by Frobenius reciprocity, $\tilde{\pi}^{G_y}$ contains a subrepresentation of $\text{ind}_{\tilde{B}_y}^{G_y} \tilde{\chi}$. But $\text{ind}_{\tilde{B}_y}^{G_y} \tilde{\chi}$ is irreducible, as $\tilde{\chi}$ is in general position, so $\tilde{\pi}^{G_y}$ contains $\text{ind}_{\tilde{B}_y}^{G_y} \tilde{\chi}$. This is the Shintani lift of $\tilde{\sigma}$ from $G_y$ to $\tilde{G}_y$ (see 2.6).

Identify $\tilde{G}_z$ with $\tilde{H} \subset \tilde{G}$. Now, by 2.6, the Shintani lift of $\tilde{\sigma}_i$ is $\text{ind}_{\tilde{B}_y}^{G_y}(w \tilde{\chi})$ for an appropriate $w \in W_{k_F}(\tilde{M}, \tilde{G})$. The argument in the preceeding paragraph, applied to $\text{ind}_{\tilde{B}_y}^{G_y}(w \chi^*) \cong \text{ind}_{\tilde{B}_y}^{G_y} \tilde{\chi}$ (where we identify $W_{k_F}(\tilde{M}, \tilde{G})$ and $W_{k_F}(\tilde{M}, \tilde{G})$), shows that $\tilde{\pi}^{G_y}$ contains $\text{ind}_{\tilde{B}_y}^{G_y}(w \chi^*)$.

6.4. L-packets and A-packets of size two. Now suppose $\Pi$ is an $L$-packet of the form $\{\pi^2(\lambda), \pi^2(\lambda)\}$ or an $A$-packet of the form $\{\pi^n(\lambda), \pi^n(\lambda)\}$ for some $\lambda \in \text{Hom}(M, \mathbb{C}^*)$ of depth zero (see case (2.7PS–2) and (4.1)). Both $\pi^2(\lambda)$ and $\pi^n(\lambda)$ are constituents of the principal series $\text{ind}_{G_y}^G\lambda$. It follows from [21] that both $\text{ind}_{G_y}^G\lambda$ and $\lambda$ have depth zero and that for any $x \in \mathcal{F}$, $(G_x, \lambda|M_0)$ is a $K$-type for both of these representations. By Proposition 5.6, $\tilde{\pi}$ is a constituent of the principal series $\text{ind}_{G_y}^G\lambda$. Therefore, as above, $(G_x, \tilde{\lambda}|_{\tilde{M}_0})$ is a $K$-type for $\tilde{\pi}$. But $\lambda|_{M_0}$ is the inflation of $\tilde{\lambda} \in \text{Hom}(\tilde{M}, \mathbb{C}^*)$, where $\tilde{\lambda} \in \text{Hom}(M, \mathbb{C}^*)$ is the character that inflates to $\lambda|_{M_0}$. This shows that the theorem is true for $\pi^2(\lambda)$ and $\pi^n(\lambda)$.

It remains to consider $\pi^*(\lambda)$ (both as an element of $\{\pi^*(\lambda), \pi^2(\lambda)\}$ and as one of $\{\pi^*(\lambda), \pi^n(\lambda)\}$). Let $\lambda_1, \lambda_2$ be the respective characters of $E^*, E^1$ determined by $\lambda$ according to (2.7.1). Suppose first that $\lambda_1|_{G_y}$ is trivial. Then Proposition 4.4 implies that $(G_y, \sigma)$ is a $K$-type for $\sigma^*(\lambda)$, where $\sigma$ is the inflation to $G_y$ of $\tau \cdot (\lambda_2 \circ \det)$. Here $\lambda_2$ is the character of $k_F^1$ determined by $\lambda_2$, and $\tau$ is the cuspidal unipotent representation of $G_y$. From §2.6, the Shintani lift of $\tau \cdot (\lambda_2 \circ \det)_{G_y}$ from
Identify (6.4.2)

where \( \tilde{\tau} \) is the unipotent representation of \( \tilde{G}_y \) that is neither the trivial nor the Steinberg representation. Let \( \tilde{\sigma} \) be the inflation of this representation to \( \tilde{G}_y \). Proposition 5.6 states that the base change lift \( \tilde{\pi} \) of \( \Pi \) is

\[
\text{ind}_{\tilde{G}^\circ}^{\tilde{G}} \tilde{\rho}',
\]

where \( \tilde{\rho}' \) is either a one-dimensional representation \( \tilde{\xi}' \) of \( \tilde{H} \) or \( \text{St}_{\tilde{H}}(\tilde{\xi}') \).

Suppose that \( \tilde{\rho}' = \tilde{\xi}' \). By Proposition 5.6,

\[
\tilde{\xi}' = (\lambda_1 \cdot \tilde{\lambda}_2) \circ \tilde{\det}_{\tilde{G}(2)} \otimes \tilde{\lambda}_2.
\]

Using Mackey’s theorem and Frobenius reciprocity, we have

\[
\begin{align*}
\text{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \text{Res}_{\tilde{G}_y} \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\xi}') &= \text{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \text{ind}_{\tilde{P} \cap \tilde{G}_y}^{\tilde{G}_y} \tilde{\xi}') \\
&= \text{Hom}_{\tilde{P} \cap \tilde{G}_y}(\tilde{\sigma}, \tilde{\xi}') \\
&= \text{Hom}_{\tilde{P} \cap \tilde{G}_y}(\tilde{\sigma}, \tilde{\xi}'^{-1}, 1),
\end{align*}
\]

(6.4.2)

where we interpret \( \tilde{\sigma} \cdot \tilde{\xi}'^{-1} \) as the product of the restriction of each factor to \( \tilde{P} \cap \tilde{G}_y \).

Identify \( \tilde{G}_y \) with \( \tilde{G} \). Since \( \lambda_1 \mid_{\tilde{G}_y} \) is trivial, \( \tilde{\xi}' \mid_{\tilde{P} \cap \tilde{G}_y} \) is the inflation to \( \tilde{P} \cap \tilde{G}_y \) of the character \( \lambda_2 \circ \tilde{\det} \) of \( \tilde{H} \). It follows that \( \tilde{\sigma} \cdot \tilde{\xi}'^{-1} \) is the restriction to \( \tilde{P} \cap \tilde{G}_y \) of the inflation to \( \tilde{G}_y \) of \( \tilde{\tau} \). Since both \( \tilde{\sigma} \cdot \tilde{\xi}'^{-1} \) and \( 1 \) are trivial on \( \tilde{G}_y^\circ \), (6.4.2) can be identified with

\[
\text{Hom}_{\tilde{P}}(\tilde{\tau}, 1),
\]

where \( \tilde{P} \) is the parabolic subgroup of \( \tilde{G}_y \) whose inverse image in \( \tilde{G}_y \) contains \( \tilde{P} \cap \tilde{G}_y \). By Frobenius reciprocity,

\[
\text{Hom}_{\tilde{P}}(\tilde{\tau}, 1) = \text{Hom}_{\tilde{G}_y}(\tilde{\tau}, \text{ind}_{\tilde{P}}^{\tilde{G}_y} 1).
\]

It is easily seen that \( \text{ind}_{\tilde{P}}^{\tilde{G}_y} 1 \) has two irreducible components: the trivial representation and \( \tilde{\tau} \). Hence

\[
\dim_{\tilde{F}} \text{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \text{Res}_{\tilde{G}_y} \text{ind}_{\tilde{P}}^{\tilde{G}_y} \tilde{\xi}') = 1.
\]

In particular, as a representation of \( \tilde{G}_y \), \( \tilde{\rho}^+ \) must contain \( \tilde{\sigma} \), as required.

Now suppose that \( \tilde{\rho}' = \text{St}_{\tilde{H}}(\tilde{\xi}') \). By Proposition 5.6, the representations \( \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\rho}' \) and \( \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\xi}' \) are the irreducible constituents of \( \text{ind}_{\tilde{B}}^{\tilde{G}} \lambda \). For all \( x \in \mathcal{F} \),

\[
\text{Res}_{\tilde{G}_y} \text{ind}_{\tilde{B}}^{\tilde{G}} \lambda = \text{ind}_{\tilde{B} \cap \tilde{G}_y}^{\tilde{G}_y} \lambda = \text{ind}_{\tilde{B} \cap \tilde{G}_y}^{\tilde{G}_y} \tilde{\lambda},
\]

which contains \( \text{ind}_{\tilde{B}}^{\tilde{G}_y} \tilde{\lambda} \), the inflation to \( \tilde{G}_y \) of the representation \( \text{ind}_{\tilde{B}}^{\tilde{G}_y} \tilde{\lambda} \). (Here \( \tilde{\lambda} \) is the character of \( M \) determined by \( \lambda \).) Moreover, \( \text{ind}_{\tilde{B}}^{\tilde{G}_y} \tilde{\lambda} \) contains two copies of the representation (6.4.1) since

\[
(\text{ind}_{\tilde{B}}^{\tilde{G}_y} \lambda) \cdot (\tilde{\lambda}_2 \circ \tilde{\det}_{\tilde{G}_y}) = \text{ind}_{\tilde{B}}^{\tilde{G}_y} 1.
\]
contains two copies of \( \tilde{\tau} \). Therefore,
\[
\dim \text{Hom}_{\tilde{G}_v}(\tilde{\sigma}, \text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}) \geq 2.
\]
Since
\[
\dim \text{Hom}_{\tilde{G}_v}(\tilde{\sigma}, \text{Res}_{\tilde{G}_v} \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\rho}) = 1,
\]
it follows that
\[
\text{Hom}_{\tilde{G}_v}(\tilde{\sigma}, \text{Res}_{\tilde{G}_v} \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\rho}) \neq 0.
\]
Hence, as above, \( \tilde{\pi} \tilde{G}_v^+ \) must contain \( \tilde{\sigma} \).

On the other hand, suppose that \( \lambda_1 \rvert_{\mathcal{O}_F^\times} \) is not trivial. Let \( \tilde{\lambda}_1 \) be the character of \( k_1^\times \) determined by \( \tilde{\lambda}_1 \circ \mathcal{N} = \lambda_1 \). Let \( \tilde{\sigma} \) be the inflation to \( G_z \) of the cuspidal representation \( \tilde{\sigma} \) of \( G_z \) with character \( -R_{\mathbb{C}}^{\mathbb{C}} \chi \), where
\[
\chi = \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_2.
\]
Then \( (G_z, \sigma) \) is a \( K \)-type for \( \pi^+(\lambda) \). By §2.6, the Shintani lift \( \tilde{\sigma} \) of \( \tilde{\sigma} \) from \( G_z \) to \( \tilde{G}_z \) is \( \text{ind}_{\tilde{B}_z}^{\tilde{G}_z} \chi \), where \( \chi = \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_2 \).

Now in both the \( L \)-packet and \( A \)-packet cases, \( \tilde{\pi} \) is a constituent of \( \text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda} \) by the proof of [23, Lemma 12.7.6]. Hence \( (\tilde{G}_x, \rvert_{\tilde{\lambda}_1 \rvert_{\tilde{M}_0}}) \) is a \( K \)-type of \( \tilde{\pi} \), where \( x \) is any point in \( \mathcal{F} \) and \( w \in W_{F'}(\tilde{M}, \tilde{G}) \). As in the case above where \( \pi \) is supercuspidal of type (2.7SC–4), it follows by Frobenius reciprocity that \( \tilde{\pi}\tilde{G}_x^+ \) contains a subrepresentation of \( \text{ind}_{\tilde{B}_x}^{\tilde{G}_x} (w \tilde{\lambda}) \), since \( \rvert_{\tilde{\lambda}_1 \rvert_{\tilde{M}_0}} \) is the inflation of the character \( w \tilde{\lambda} \) of \( \tilde{M} \) (where we have identified \( W_{F'}(\tilde{M}, \tilde{G}) \) and \( W_{k_{F'}}(\tilde{M}, \tilde{G}) \)). Using the fact that \( \lambda_1 \rvert_{\mathcal{O}_F^\times} \) is trivial, one finds that
\[
\tilde{\lambda} = \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_1 \tilde{\lambda}_2,
\]
so for an appropriate \( w \),
\[
w \tilde{\lambda} = \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_2.
\]
Thus \( \text{ind}_{\tilde{B}_x}^{\tilde{G}_x} (w \tilde{\lambda}) = \tilde{\sigma} \), so \( \tilde{\pi}^{\tilde{G}_x^+} \) contains the irreducible representation \( \tilde{\sigma} \).

7. On Induced Characters of Nonconnected Groups

Let \( G \) now denote the group of rational points of a reductive group defined over a nonarchimedean local field. In particular, we do not assume that \( G \) is connected. We will, however, assume that \( G \) is a semidirect product of its connected component \( G^0 \) and its component group \( \Gamma \), and that \( G \) has a \( \Gamma \)-invariant special parahoric subgroup. Suppose \( H \) is an open subgroup of \( G \) that is compact modulo the center of \( G \). Let \( \rho \) denote an irreducible, smooth representation of \( H \), and let \( \pi \) denote the compactly induced representation \( \text{ind}_H^G \rho \) of \( G \). Let \( K \) denote a compact open subgroup of \( G \).

**Proposition 7.1.** For \( g \in G^\text{reg} \),
\[
\theta_x(g) = \sum_{a \in K \backslash G/H} \left( \sum_{b \in K a H/H} \theta_{\rho}(b^{-1} g b) \right),
\]
where \( \theta_{\rho} \) is extended to \( G \) by zero. For each \( g \), all but finitely many terms of the inner sum vanish.

**Proof.** This is identical to the proof of Theorem A.14 of [5]. \( \square \)
REFERENCES


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